KEMENTERIAN PENDIDIKAN MALAYSIA

DUAL LANGUAGE PROGRAMME

ADDITIONAL MATHEMATICS

FORM 5



KURIKULUM STANDARD SEKOLAH MENENGAH
KSSM
KEMENTERIAN PENDIDIKAN MALAYSIA

KEMENTERIAN PENDIDIKAN MALAYSIA



RUKUN NEGARA

Bahawasanya Negara Kita Malaysia
mendukung cita-cita hendak;

Mencapai perpaduan yang lebih erat dalam kalangan
seluruh masyarakatnya;

Memelihara satu cara hidup demokrasi;

Mencipta satu masyarakat yang adil di mana kemakmuran negara
akan dapat dinikmati bersama secara adil dan saksama;

Menjamin satu cara yang liberal terhadap
tradisi-tradisi kebudayaannya yang kaya dan pelbagai corak;

Membina satu masyarakat progresif yang akan menggunakan
sains dan teknologi moden;

MAKA KAMI, rakyat Malaysia,
berikrar akan menumpukan
seluruh tenaga dan usaha kami untuk mencapai cita-cita tersebut
berdasarkan prinsip-prinsip yang berikut:

KEPERCAYAAN KEPADA TUHAN
KESETIAAN KEPADA RAJA DAN NEGARA
KELUHURAN PERLEMBAGAAN
KEDAULATAN UNDANG-UNDANG
KESOPANAN DAN KESUSILAAN

KEMENTERIAN PENDIDIKAN MALAYSIA

(Sumber: Jabatan Penerangan, Kementerian Komunikasi dan Multimedia Malaysia)

KURIKULUM STANDARD SEKOLAH MENENGAH

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2020

HAK MILIK JABATAN PENDIDIKAN MALAYSIA

KEMENTERIAN PENDIDIKAN MALAYSIA

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Introduction

The Form 5 Additional Mathematics KSSM textbook is written based on *Dokumen Standard Kurikulum dan Pentaksiran* (*DSKP*) Additional Mathematics Form 5 prepared by the Ministry of Education.

The book is published to produce pupils who have 21st century skills by applying Higher Order Thinking Skills (HOTS), information and communication skills, thinking and problem-solving skills and interpersonal and self-direction skills so that they can compete globally. Pupils who master high-level thinking skills are able to apply the knowledge, skills and values to reason and reflect in solving problems, making decisions, innovating and creating new things.

Cross-curricular elements such as the use of proper language of instruction, environmental sustainability, moral values, science and technology, patriotism, creativity and innovation, entrepreneurship, information and communication technology, global sustainability and financial education are applied extensively in the production of the content of this textbook. In addition, it is given the STEM approach so that pupils have the opportunity to integrate knowledge, skills and values in science, technology, engineering and mathematics. The book also emphasises on the application of computational thinking (CT).

SPECIAL FEATURES OF THIS BOOK AND THEIR FUNCTIONS

Discovery Activity 1 Individual Discovery Activity 1 Pair Discovery Activity 1 Group	Activities involving the pupils individually, in pairs or in groups that encourage pupils to actively participate in the learning process
Self-Exercise 1.1	Exposes pupils to questions that evaluate their understanding of the concepts learned
Formative Exercise 1.1	Contains questions to determine pupil's mastery of the topic
MATHEMATICAL APPLICATIONS	Provides troubleshooting questions along with the work steps involving real-life situations
Recall	Assists pupils in recalling information learned
	Asks questions that require pupils to think creatively and test pupil's mastery
Information Corner	Provides additional information for pupils to understand the topic further
HISTORY GALLERY	Sheds light on the history of mathematics and mathematical figures' contribution
DISCUSSION	Contains activities that require discussion among pupils

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Calculator Literate	Explains how to use a scientific calculator in mathematics calculations
Technology Click	Gives exposure to pupils on the application of technology in the learning of mathematics
QR Access	Exposes pupils to mobile devices for scanning the QR code
Excellent Tip	Gives tips related to topics for pupil use
Alternative Method	Provides alternative solutions to certain questions
CT	Discovery activity involving computational thinking that includes the concepts of logical reasoning, algorithms, pattern recognition, scaling and evaluation
PBL	Project-based Learning allows pupils to apply mathematical knowledge and skills in solving daily life problems
REFLECTION CORNER	Conclusions on what have been studied in the chapter
Summative Exercise	Questions in the forms of LOTS and HOTS to determine the performance level of pupils
	Contains HOTS questions to test pupils' higher-order thinking skills
21st cl	The 21st century learning concept is applied to increase the pupils' level of understanding
1.3.1	Represents the learning standards for each chapter
PL 1 PL 2 PL 3 PL 4 PL 5 PL 6	Includes the performance level for each question
STEM	Discovery activity that applies the concepts of science, technology, engineering and mathematics

Scanning Guide AR (Augmented Reality) for the Interactive Three-Dimensional Animation

Scan the QR code to download the application.

Use the application to scan the pages with icons AR (pages 105 and 106).



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Formulae

Arc length, $s = r\theta$

Area of sector, $A = \frac{1}{2}r^2\theta$

Heron formula $= \sqrt{s(s-a)(s-b)(s-c)}$,

$$s = \frac{a+b+c}{2}$$

$${}^nP_r = \frac{n!}{(n-r)!}$$

$${}^nC_r = \frac{n!}{(n-r)!r!}$$

Identical formula, $P = \frac{n!}{a!b!c!\ldots}$

$$y = uv, \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$y = \frac{u}{v}, \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$P(X = r) = {}^nC_r p^r q^{n-r}$, $p + q = 1$

Mean, $\mu = np$

$\sigma = \sqrt{npq}$

$$Z = \frac{X - \mu}{\sigma}$$

Area under a curve

$= \int_a^b y\,dx$ or

$= \int_a^b x\,dy$

Volume of revolution

$= \int_a^b \pi y^2\,dx$ or

$= \int_a^b \pi x^2\,dy$

$\sin^2 A + \cos^2 A = 1$

$\sec^2 A = 1 + \tan^2 A$

$\operatorname{cosec}^2 A = 1 + \cot^2 A$

$\sin 2A = 2 \sin A \cos A$

$\cos 2A = \cos^2 A - \sin^2 A$

$= 2\cos^2 A - 1$

$= 1 - 2\sin^2 A$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

bit.ly/2ttRUrS

Download the free application QR code scan from Google Play, App Store or other applications to your smart mobile devices. Scan the QR code with the application or visit the website listed on the left via the QR code to download the PDF file, GeoGebra and full answers. Then, save the file downloaded for offline use.

KEMENTERIAN PENDIDIKAN MALAYSIA

CHAPTER 1

CIRCULAR MEASURE



What will be learnt?

- Radian
- Arc Length of a Circle
- Area of Sector of a Circle
- Application of Circular Measures

List of Learning Standards

bit.ly/2QDBAxI

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In the 21st century, technology and innovation are evolving at a very rapid pace. Innovatively designed buildings can increase the prestige of a country. An architect can design very unique and beautiful buildings with special software together with his or her creative and innovative abilities. How can the buildings be structurally sound and yet retain their dynamic designs? What does an architect need to know to design a major segment of a circular building like the one shown in the picture?



Video on round building architecture

bit.ly/35E1wh1

Info Corner

Euclid (325-265 BC) was a Greek mathematician from Alexandria. He is well known for his work 'The Elements', a study in the field of geometry.

Geometrical mathematics is concerned with sizes, shapes and relative positions in diagrams and space characteristics.

For more info:

bit.ly/35KqImk

Significance of the Chapter

- An air traffic controller uses his skills in reading and interpreting radar at the air traffic control centre to guide planes to fly safely without any collision in the air, which may result in injury and death.
- Odometer in a vehicle records the total mileage covered from the beginning to the end of the journey by using the circumference of the tyre and the number of rotations of the tyre.

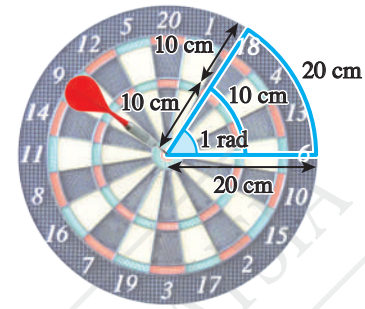
Keywords

Radian	*Radian*
Degree	*Darjah*
Centre of circle	*Pusat bulatan*
Radius	*Jejari*
Segment	*Tembereng*
Sector	*Sektor*
Perimeter	*Perimeter*
Arc length	*Panjang lengkok*
Area of sector	*Luas sektor*

KEMENTERIAN PENDIDIKAN MALAYSIA

1.1 Radian

The diagram on the right shows two sectors marked on a dartboard with radii 10 cm and 20 cm and their respective arc lengths of 10 cm and 20 cm. Since each arc length is the same length as its radius, the angle subtended at the centre of the circle is defined as 1 radian. That is, the size of the angle subtended by both arcs at the centre of the circle should be the same.



What can you say about the measurement of the angle of 1 radian?

Relating angle measurement in radians and degrees

In circular measures, the normal unit used to measure angles is in degrees. However, in some mathematical disciplines, circular measures in degrees are less suitable. Therefore, a new unit called the radian is introduced to measure the size of an angle.

Information Corner

- "Rad" stands for "Radian".
- 1 rad can be written as 1^r or 1^c.

The activity below will explain the definition of one radian and at the same time relates angles measured in degrees to those measured in radians.

Discovery Activity 1 Group STEM CT

Aim: To explain the definition of one radian and then relate angles measured in radians to angles in degrees

bit.ly/2R1JvEe

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Each group is required to do each of the following activities by recording the angle subtended at the centre of the circle.

   Drag slider *a* such that the length of the arc, *s* is the same length as the radius of the circle, *r*.

   Drag slider *a* such that the length of the arc, *s* is twice the length of the radius of the circle, *r*.

   Drag slider *a* such that the length of the arc, *s* is three times the length of the radius of the circle, *r*.

   Drag slider *a* such that the length of the arc, *s* is the length of the semicircle.

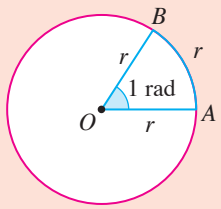
   Drag slider *a* such that the arc length, *s* is the length of the circumference of the circle.

3. Based on the results obtained, define an angle of 1 radian. Then, relate radians to degrees for the angle subtended at the centre of the circle.
4. From this relation, estimate an angle of 1 radian in degrees and an angle of 1° in radians. Discuss your answer.

KEMENTERIAN PENDIDIKAN MALAYSIA

From the Discovery Activity 1, the definition of one radian is as follows:

One radian is the measure of an angle subtended at the centre of a circle by an arc whose length is the same as the radius of the circle.



In general, for a circle with centre O and radius r units:

- If the arc length $AB = r$, then $\angle AOB = 1$ radian.
- If the arc length $AB = 2r$, then $\angle AOB = 2$ radians.
- If the arc length $AB = 3r$, then $\angle AOB = 3$ radians.
- If the arc length $AB = \pi r$, then $\angle AOB = \pi$ radians.
- If the arc length $AB = 2\pi r$, then $\angle AOB = 2\pi$ radians.

Note that when the arc length AB is $2\pi r$, it means that OA has made a complete rotation or OA has rotated through 360°. Hence, we can relate radians to degrees as follows.

$$2\pi \text{ rad} = 360°$$

$$\pi \text{ rad} = 180°$$

Hence, when $\pi = 3.142$,

$$1 \text{ rad} = \frac{180°}{\pi} \approx 57.29°$$

and $$1° = \frac{\pi}{180°} \approx 0.01746 \text{ rad}$$

HISTORY GALLERY



Gottfried Wilhelm Leibniz was a brilliant German mathematician who introduced a method to calculate the value of $\pi = 3.142$ without using a circle. He also proved that $\frac{\pi}{4}$ can be obtained by using the following formula.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

DISCUSSION

1 radian is smaller than 60°. What are the advantages of using angles in radians compared to angles in degrees? Discuss.

Example 1

Convert each of the following angles into degrees. [Use $\pi = 3.142$]

(a) $\frac{2}{5}\pi$ rad (b) 2.25 rad

Solution

(a) π rad = 180°

$$\frac{2}{5}\pi \text{ rad} = \frac{2}{5}\pi \times \frac{180°}{\pi}$$
$$= \frac{2}{5} \times 180°$$
$$= 72°$$

(b) π rad = 180°

$$2.25 \text{ rad} = 2.25 \times \frac{180°}{\pi}$$
$$= 2.25 \times \frac{180°}{3.142}$$
$$= 128° 54'$$

Calculator Literate

To find the solution for Example 1(b) using a scientific calculator.

1. Press
   2 . 2 5
2. Press
   OPTN 2 2 =
3. The screen will display
   2.25r
   128.9155039

Example 2

(a) Convert 40° and 150° into radians, in terms of π.
(b) Convert 110° 30′ and 320° into radians.
[Use π = 3.142]

Solution

(a) $180° = \pi$ rad

$40° = 40° \times \frac{\pi}{180°}$

$= \frac{2}{9}\pi$ rad

$150° = 150° \times \frac{\pi}{180°}$

$= \frac{5}{6}\pi$ rad

(b) $180° = \pi$ rad

$110°\ 30' = 110°\ 30' \times \frac{\pi}{180°}$

$= 110°\ 30' \times \frac{3.142}{180°}$

$= 1.929$ rad

$320° = 320° \times \frac{\pi}{180°}$

$= 320° \times \frac{3.142}{180°}$

$= 5.586$ rad

Excellent Tip

Special angles:

Angle in degree	Angle in radian
0°	0
30°	$\frac{\pi}{6}$
36°	$\frac{\pi}{5}$
45°	$\frac{\pi}{4}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
180°	π
270°	$\frac{3}{2}\pi$
360°	2π

Self-Exercise 1.1

1. Convert each of the following angles into degrees. [Use π = 3.142]

(a) $\frac{\pi}{8}$ rad (b) $\frac{3}{4}\pi$ rad (c) 0.5 rad (d) 1.04 rad

2. Convert each of the following angles into radians, in terms of π.

(a) 18° (b) 120° (c) 225° (d) 300°

Formative Exercise 1.1

Quiz bit.ly/2QGcIWr

1. Convert each of the following angles into degrees. [Use π = 3.142]

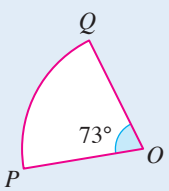
(a) $\frac{7}{12}\pi$ rad (b) $1\frac{1}{3}\pi$ rad (c) 2 rad (d) 4.8 rad

2. Convert each of the following angles into radians. Give answers correct to three decimal places. [Use π = 3.142]

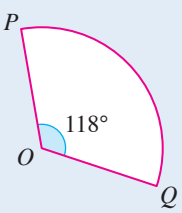
(a) 76° (b) 139° (c) 202.5° (d) 320° 10′

3. In each of the following diagrams, *POQ* is a sector of a circle with centre *O*. Convert each of the angles *POQ* into radians. [Use π = 3.142]

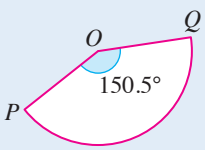
(a)



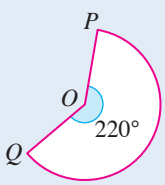
(b)



(c)



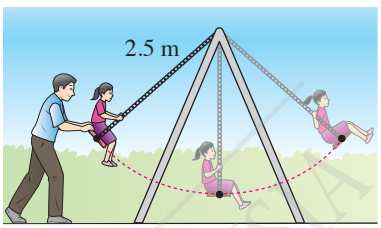
(d)



KEMENTERIAN PENDIDIKAN MALAYSIA

1.2 Arc Length of a Circle

The diagram on the right shows a little girl on a swing. The swing sweeps through 1.7 radians and makes an arc of a circle. What is the arc length made by the little girl on that swing?



What formula can be used to solve this problem?

Determining the arc length, radius and the angle subtended at the centre of a circle

Discovery Activity 2

Group | 21st cl | STEM | CT

Aim: To derive the formula for the arc length of a circle with centre O

Steps:

ggbm.at/haatecxq

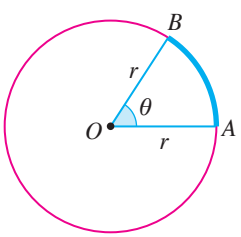
1. Scan the QR code on the right or visit the link below it.
2. Drag the point A or B along the circumference of the circle to change the arc length AB.
3. Note the arc length AB and the angle AOB in degrees subtended at the centre of the circle when the point A or B changes.
4. What do you observe concerning the value of the ratios $\frac{\text{Minor arc length } AB}{\text{Circumference}}$ and $\frac{\text{Angle } AOB}{360°}$? Are the ratios the same?
5. Drag the slider L to vary the size of the circle. Are the two ratios from step 4 above still the same?
6. Then, derive a formula to determine the minor arc length of a circle.
7. Record all the results from the members of your group on a piece of paper.
8. Each group presents their findings to the class and finally come up with a conclusion concerning this activity.

From Discovery Activity 2, it is found that the arc length of a circle is proportional to the angle subtended at the centre of the circle.

$$\frac{\text{Minor arc length } AB}{\angle AOB} = \frac{\text{Circumference}}{360°}$$

$$\frac{\text{Minor arc length } AB}{\theta} = \frac{2\pi r}{360°}$$

$$\text{Minor arc length } AB = \frac{2\pi r}{360°} \times \theta$$



where θ is the angle in degrees subtended at the centre of the circle, O whose radius is r units.

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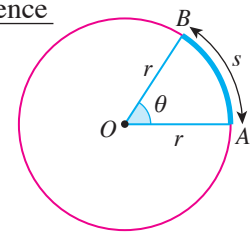
However, if $\angle AOB$ is measured in radians,

$$\frac{\text{Minor arc length } AB}{\theta} = \frac{\text{Circumference}}{2\pi}$$

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi}$$

$$s = \frac{2\pi r}{2\pi} \times \theta$$

$$s = r\theta$$



Information Corner

The symbol θ is read as "téta", which is the eighth letter in the Greek alphabet and it is often used to represent an angle.

In general,

$$s = r\theta$$

where s is the arc length of the circle with radius r units and θ radian is the angle subtended by the arc at the centre of the circle, O.

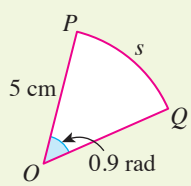
DISCUSSION

From the definition of radian, can you derive the formula $s = r\theta$?

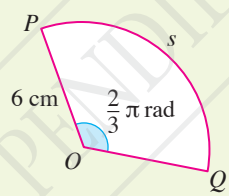
Example 3

Find the arc length, s for each of the following sectors POQ with centre O.
[Use $\pi = 3.142$]

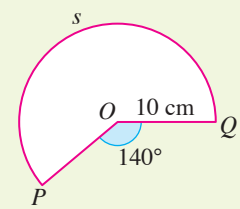
(a)



(b)



(c)



Solution

(a) Arc length, $s = r\theta$

$s = 5 \times 0.9$

$s = 4.5$ cm

(b) Arc length, $s = r\theta$

$s = 6 \times \frac{2}{3}\pi$

$s = 4\pi$

$s = 4(3.142)$

$s = 12.57$ cm

(c) Reflex angle POQ in radians

$= (360° - 140°) \times \frac{\pi}{180°}$

$= 220° \times \frac{3.142}{180°}$

$= 3.84$ rad

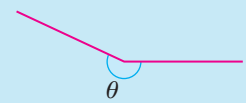
Arc length, $s = r\theta$

$s = 10 \times 3.84$

$s = 38.4$ cm

Recall

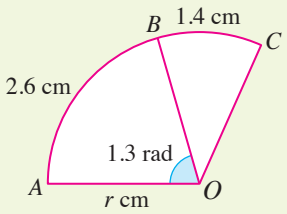
The angle size of a reflex angle is $180° < \theta < 360°$.



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Example 4

The diagram on the right shows a part of a circle with centre O and a radius of r cm. Given that $\angle AOB = 1.3$ rad and the arc lengths AB and BC are 2.6 cm and 1.4 cm respectively, calculate
(a) the value of r,
(b) $\angle BOC$, in radians.

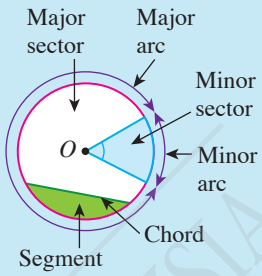


Solution

(a) For sector AOB,
$s = 2.6$ cm and
$\theta = 1.3$ rad.
Thus, $s = r\theta$
$r = \frac{s}{\theta}$
$r = \frac{2.6}{1.3}$
$r = 2$ cm

(b) For sector BOC,
$s = 1.4$ cm and $r = 2$ cm.
Hence, $s = r\theta$
$\theta = \frac{s}{r}$
$\theta = \frac{1.4}{2}$
$\theta = 0.7$ rad
Thus, $\angle BOC = 0.7$ rad.

Recall



QR Access

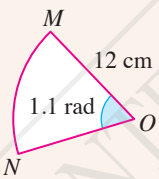
Recognising a circle

bit.ly/37Tju0u

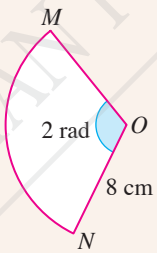
Self-Exercise 1.2

1. Find the arc length MN, in cm, for each of the following sectors MON with centre O.
[Use $\pi = 3.142$]

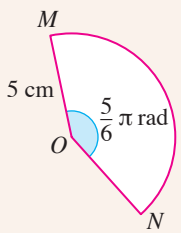
(a)



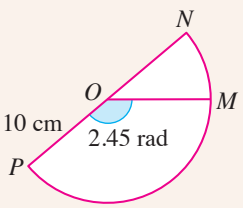
(b)



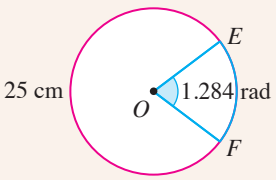
(c)



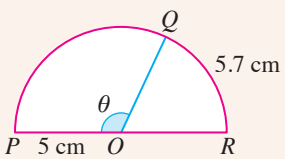
(d)



2. The diagram on the right shows a circle with centre O. Given that the major arc length EF is 25 cm and $\angle EOF = 1.284$ rad, find
(a) the radius, in cm, of the circle,
(b) the minor arc length EF, in cm.
[Use $\pi = 3.142$]



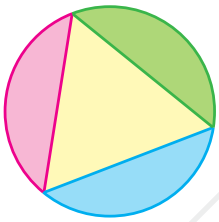
3. The diagram on the right shows semicircle $OPQR$ with a radius of 5 cm. Given that the arc length QR is 5.7 cm, calculate
(a) the value of θ, in radians,
(b) the arc length PQ, in cm.
[Use $\pi = 3.142$]



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Determining the perimeter of segment of a circle

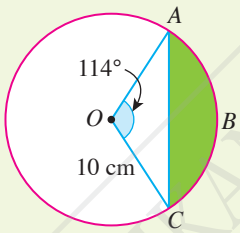
The coloured region of the rim of the bicycle tyre with a radius of 31 cm in the diagram consists of three identical segments of a circle. The perimeter for one of the segments is the sum of all its sides.



With the use of the arc length formula $s = r\theta$ and other suitable rules or formulae, can you find the perimeter of any one of the segments?

Example 5

The diagram on the right shows a circle with centre O and a radius of 10 cm. The chord AC subtends an angle of 114° at the centre of the circle. Calculate the perimeter of the shaded segment ABC. [Use $\pi = 3.142$]



Solution

Since $180° = \pi$ rad, we have

$$114° = 114° \times \frac{\pi}{180°}$$
$$= 1.990 \text{ rad}$$

Arc length $ABC = r\theta$
$= 10 \times 1.990$
$= 19.90$ cm

With cosine rule, the length of chord AC is

$$AC^2 = 10^2 + 10^2 - 2(10)(10)\cos 114°$$
$$AC = \sqrt{200 - 200\cos 114°}$$
$$= 16.77 \text{ cm}$$

Thus, the perimeter of the shaded segment $ABC = 19.90 + 16.77$
$= 36.67$ cm

Alternative Method

To find the chord AC, draw a perpendicular line, OD from O to chord AC.
In ΔCOD,

$$\angle COD = \frac{114°}{2}$$
$$= 57°$$
$$\sin \angle COD = \frac{CD}{OC}$$

Hence, $CD = OC \sin \angle COD$
$= 10 \sin 57°$
$= 8.3867$ cm

Thus, $AC = 2CD$
$= 2(8.3867)$
$= 16.77$ cm

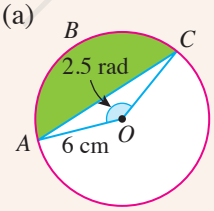
Flash Quiz

Can the length of AC be obtained using sine rule, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$?

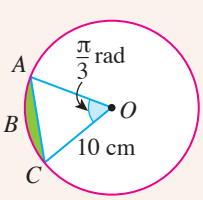
Self-Exercise 1.3

1. For each of the following circles with centre O, find the perimeter, in cm, of the shaded segment ABC. [Use $\pi = 3.142$]

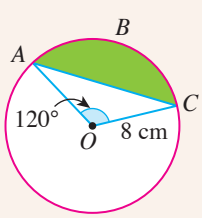
(a)



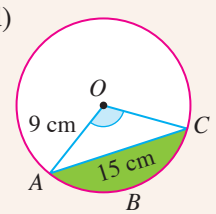
(b)



(c)

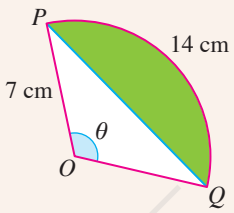


(d)



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2. The diagram on the right shows a sector with centre O and a radius of 7 cm. Given that the arc length PQ is 14 cm, find
(a) the angle θ, in degrees,
(b) the perimeter of the shaded segment, in cm.

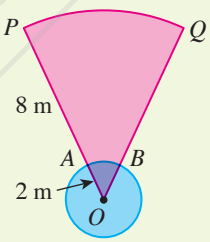


Solving problems involving arc lengths

With the knowledge and skills of converting angles from degrees to radians and vice versa, as well as the arc length formula, $s = r\theta$ and other suitable rules, we can solve many problems in our daily lives involving arc length of a circle.

Example 6 MATHEMATICAL APPLICATIONS

The diagram on the right shows the region for the shot put event drawn on a school field. The region is made up of two sectors from two circles, AOB and POQ, both with centre O. Given that $\angle AOB = \angle POQ = 50°$, $OA = 2$ m and $AP = 8$ m, calculate the perimeter of the coloured region $ABQP$, in m. [Use $\pi = 3.142$]



Solution

1 . Understanding the problem

- The shot put region consists of two sectors AOB and POQ from two circles, both with centre O.
- The sector AOB has a radius of 2 m, $AP = 8$ m and $\angle AOB = \angle POQ = 50°$.

2 . Planning the strategy

- Convert 50° into radians and use the formula $s = r\theta$ to find the arc lengths AB and PQ.
- The perimeter of the shaded region $ABQP$ can be obtained by adding all the sides enclosing it.

3 . Implementing the strategy

$180° = \pi$ rad

$50° = 50° \times \dfrac{3.142}{180°}$

$= 0.873$ rad

Arc length AB, $s = r\theta$

$s = 2(0.873)$

$s = 1.746$ m

Arc length PQ, $s = r\theta$

$s = 10(0.873)$

$s = 8.73$ m

Thus, the perimeter of the shaded region $ABQP$

= arc length AB + BQ + arc length PQ + AP

= 1.746 + 8 + 8.73 + 8

= 26.48 m

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4 . Check and reflect

Arc length $AB = \frac{50°}{360°}(2)(3.142)(2)$
$= 1.746$ m

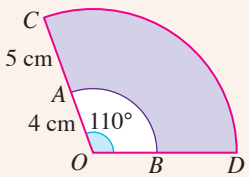
Arc length $PQ = \frac{50°}{360°}(2)(3.142)(10)$
$= 8.73$ m

Thus, the perimeter of the shaded region $ABQP$
= arc length AB + BQ + arc length PQ + AP
= 1.746 + 8 + 8.73 + 8
= 26.48 m

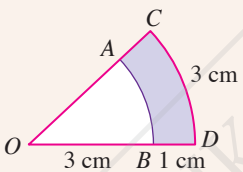
Self-Exercise 1.4

1. In each of the following diagrams, calculate the perimeter, in cm, of the shaded region.

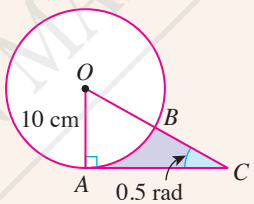
(a)



(b)

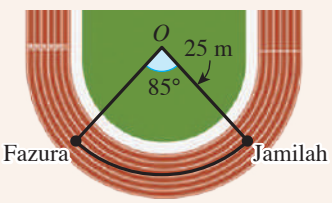


(c)



2. The city of Washington in United States of America and the city of Lima in Peru lie on the same longitude but are on latitudes 38.88° N and 12.04° S respectively. Given that the earth is a sphere with a radius of 6 371 km, estimate the distance, in km, between the two cities.

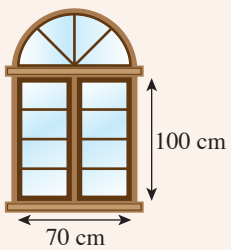
3. The diagram on the right shows a part of a running track which is semicircular in shape. Fazura wants to pass the baton to Jamilah, who is waiting at 85° from her. How far must Fazura run in order to pass the baton to Jamilah?



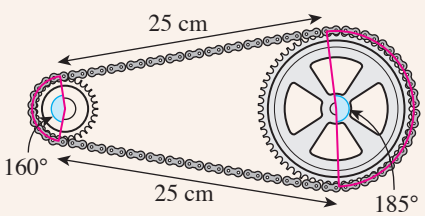
4. The diagram on the right shows a window which consists of a rectangle and a semicircle. The width and height of the rectangle are 70 cm and 100 cm respectively. Find

(a) the arc length of the semicircle of the window, in cm,

(b) the perimeter of the whole window, in cm.



5. The diagram shows the chain linking the front and back cranks of a bicycle. It is given that the circumference of the front and back cranks are 50.8 cm and 30.5 cm respectively. Calculate the length of the bicycle chain, in cm.

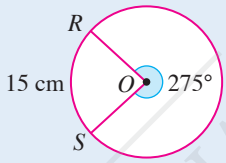


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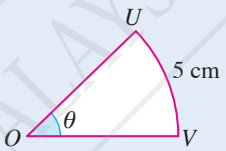
Formative Exercise 1.2

Quiz bit.ly/39W9p4V

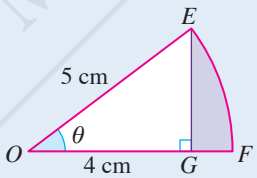
1. The diagram on the right shows a circle with centre O. The minor arc length RS is 15 cm and the angle of the major sector ROS is 275°. Find
(a) the angle subtended by the minor sector ROS, in radians,
(b) the radius of the circle, in cm.



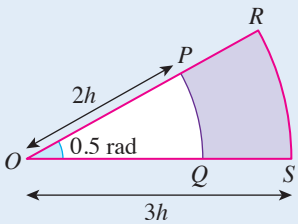
2. The diagram on the right shows sector UOV with centre O. Given that the arc length UV is 5 cm and the perimeter of sector UOV is 18 cm, find the value of θ, in radians.



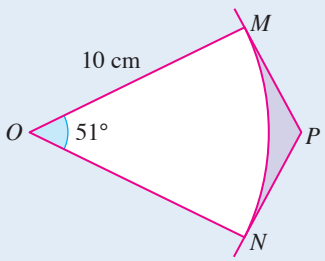
3. The diagram on the right shows sector EOF of a circle with centre O. Given that $OG = 4$ cm and $OE = 5$ cm, find
(a) the value of θ, in radians,
(b) the perimeter of the shaded region, in cm.



4. The diagram on the right shows two sectors, OPQ and ORS, with centre O and radii $2h$ cm and $3h$ cm respectively. Given that $\angle POQ = 0.5$ radian and the perimeter of the shaded region $PQSR$ is 18 cm, find
(a) the value of h, in cm,
(b) the difference in length, in cm, between the arc lengths of RS and PQ.



5. The diagram on the right shows a part of a circle with centre O and a radius of 10 cm. Tangents to the circle at point M and point N meet at P and $\angle MON = 51°$. Calculate
(a) arc length MN, in cm,
(b) the perimeter of the shaded region, in cm.



6. A wall clock has a pendulum with a length of 36 cm. If it swings through an angle of 21°, find the total distance covered by the pendulum in one complete oscillation, in cm.

7. The diagram on the right shows the measurement of a car tyre. What is the distance travelled, in m, if it makes
(a) 50 complete oscillations?
(b) 1 000 complete oscillations?
[Use $\pi = 3.142$]



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1.3 Area of Sector of a Circle

A pizza with a radius of 10 cm is cut into 10 equal pieces. Can you estimate the surface area of each piece?

What formula can be used to solve this problem?



Determining the area of sector, radius and the angle subtended at the centre of a circle

The area of a sector of a circle is the region bounded by the arc length and the two radii. The following discovery activity shows how to derive the formula for the area of a sector of a circle by using the dynamic GeoGebra geometry software.

Discovery Activity 3

Group | 21st cl | STEM | CT

Aim: To derive the formula for the area of a sector of a circle with centre O

Steps:

ggbm.at/rdpf3rx9

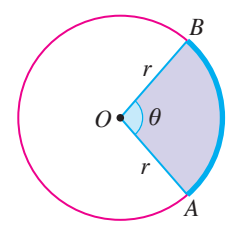
1. Scan the QR code on the right or visit the link below it.
2. Drag the point A or B along the circumference to change the area of the minor sector AOB.
3. Pay attention to the area of the sector AOB and the angle AOB in degrees subtended at the centre of the circle when the point A or B changes.
4. What are your observations on the values of the ratios $\frac{\text{Area of minor sector } AOB}{\text{Area of the circle}}$ and $\frac{\text{Angle } AOB}{360°}$? Are the values of the two ratios the same?
5. Drag the slider L to change the size of the circle. Are the two above ratios still the same?
6. Subsequently, derive the formula for the area of a minor sector of a circle. Record all the values from the members of your group on a piece of paper.
7. Each group presents their findings to the class and subsequently draws a conclusion from this activity.
8. Members from other groups can give feedback on the presentations given.

From Discovery Activity 3, we found that:

$$\frac{\text{Area of minor sector } AOB}{\angle AOB} = \frac{\text{Area of the circle}}{360°}$$

$$\frac{\text{Area of minor sector } AOB}{\theta} = \frac{\pi r^2}{360°}$$

$$\text{Area of minor sector } AOB = \frac{\pi r^2}{360°} \times \theta$$



where θ is the angle in degrees subtended at the centre of the circle, O whose radius is r units.

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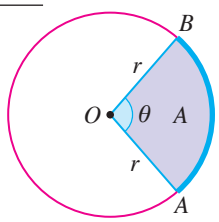
However, if $\angle AOB = \theta$ is measured in radians,

$$\frac{\text{Area of minor sector } AOB}{\theta} = \frac{\text{Area of the circle}}{2\pi}$$

$$\frac{A}{\theta} = \frac{\pi r^2}{2\pi}$$

$$A = \frac{\pi r^2}{2\pi} \times \theta$$

$$A = \frac{1}{2} r^2 \theta$$



QR Access

Alternative method to derive the formula of area of a sector of a circle, $A = \frac{1}{2} r^2 \theta$.

bit.ly/39YqDOT

In general,

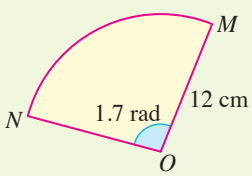
$$A = \frac{1}{2} r^2 \theta$$

where A is the area of a sector of the circle with radius r units and θ radian is the angle subtended by the sector at the centre O of the circle.

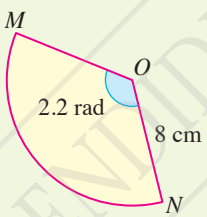
Example 7

Find the area of sector, A for each sector MON with centre O. [Use $\pi = 3.142$]

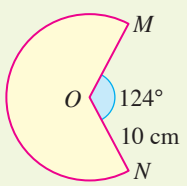
(a)



(b)



(c)



Solution

(a) Area of the sector, $A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2}(12)^2(1.7)$$

$$A = \frac{1}{2}(144)(1.7)$$

$$A = 122.4 \text{ cm}^2$$

(b) Area of the sector, $A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2}(8)^2(2.2)$$

$$A = \frac{1}{2}(64)(2.2)$$

$$A = 70.40 \text{ cm}^2$$

(c) Reflex angle MON in radians

$$= (360° - 124°) \times \frac{\pi}{180°}$$

$$= 236° \times \frac{3.142}{180°}$$

$$= 4.12 \text{ rad}$$

Area of the sector, $A = \frac{1}{2} r^2 \theta$

$$A = \frac{1}{2}(10)^2(4.12)$$

$$A = \frac{1}{2}(100)(4.12)$$

$$A = 206 \text{ cm}^2$$

Information Corner

Area of a sector, A is $A = \frac{1}{2} r^2 \theta$, where θ is the angle in radians. Since $s = r\theta$, we obtained:

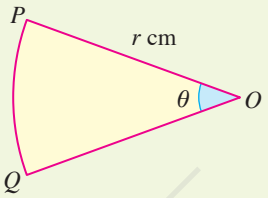
$$A = \frac{1}{2} r(r\theta)$$

$$A = \frac{1}{2} rs$$

KEMENTERIAN PENDIDIKAN MALAYSIA

Example 8

The diagram on the right shows a sector POQ which subtends an angle of θ radians and has a radius of r cm. Given that the area of the sector POQ is 35 cm^2, find
(a) the value of r if $\theta = 0.7$ rad,
(b) the value of θ if the radius is 11 cm.



Solution

(a) Area of sector POQ = 35 cm^2

$$\frac{1}{2}r^2\theta = 35$$
$$\frac{1}{2}r^2(0.7) = 35$$
$$r^2 = \frac{35 \times 2}{0.7}$$
$$r^2 = 100$$
$$r = \sqrt{100}$$
$$r = 10 \text{ cm}$$

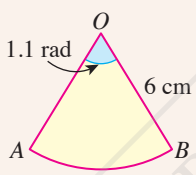
(b) Area of sector POQ = 35 cm^2

$$\frac{1}{2}r^2\theta = 35$$
$$\frac{1}{2}(11)^2\theta = 35$$
$$\frac{1}{2}(121)\theta = 35$$
$$\theta = \frac{35 \times 2}{121}$$
$$\theta = 0.5785 \text{ rad}$$

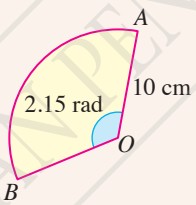
Self-Exercise 1.5

1. For each of the following sectors of circles AOB with centre O, determine the area, in cm^2. [Use $\pi = 3.142$]

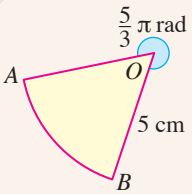
(a)



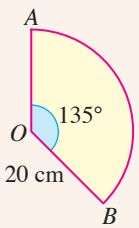
(b)



(c)

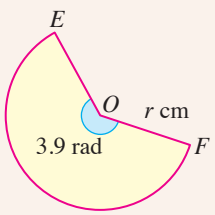


(d)

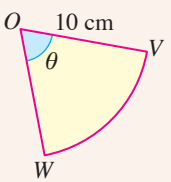


2. A sector of a circle has a radius of 5 cm and a perimeter of 16 cm. Find the area of the sector, in cm^2.

3. The diagram on the right shows a major sector EOF with centre O, a radius of r cm and an area of 195 cm^2. Calculate
(a) the value of r, in cm,
(b) the major arc length EF, in cm,
(c) the perimeter of the major sector EOF, in cm.



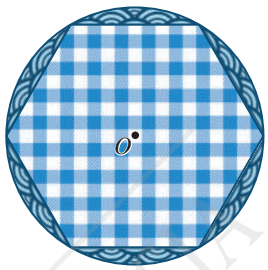
4. The diagram on the right shows a sector VOW with centre O and a radius of 10 cm. Given that the area of the sector is 60 cm^2, calculate
(a) the value of θ, in radians,
(b) the arc length VW, in cm,
(c) the perimeter of sector VOW, in cm.



KEMENTERIAN PENDIDIKAN MALAYSIA

Determining the area of segment of a circle

The diagram on the right shows a circular piece of a table cloth with centre O with an inscribed hexagon pattern. The laces sewn around the hexagon form segments on the table cloth. What information is needed to find the area of each lace?

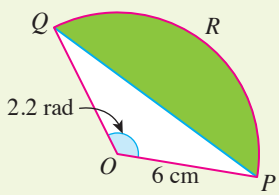


By using the formula of a sector, $A = \frac{1}{2}r^2\theta$ and other suitable formulae, this problem can be solved easily and fast.

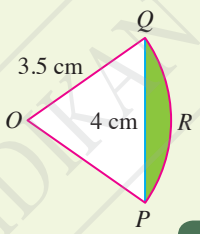
Example 9

For each of the following given sectors POQ with centre O, find the area of the segment PRQ, in cm^2.
[Use $\pi = 3.142$]

(a)



(b)



Solution

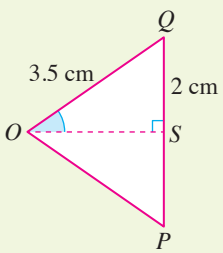
(a) $2.2 \text{ rad} = 2.2 \times \frac{180°}{3.142}$
$= 126° 2'$

Area of sector $POQ = \frac{1}{2}r^2\theta$
$= \frac{1}{2}(6)^2(2.2)$
$= 39.60 \text{ cm}^2$

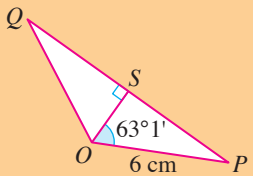
Area of $\Delta POQ = \frac{1}{2}(OP)(OQ)\sin \angle POQ$
$= \frac{1}{2}(6)(6)\sin 126° 2'$
$= 14.56 \text{ cm}^2$

Area of the segment $PRQ = 39.60 - 14.56$
$= 25.04 \text{ cm}^2$

(b) In ΔQOP, $\sin \angle QOS = \frac{QS}{OQ}$
$= \frac{2}{3.5}$
$\angle QOS = 34° 51'$



Alternative Method



In ΔPOQ,
$\angle POS = \frac{126° 2'}{2}$
$= 63° 1'$

$\sin 63° 1' = \frac{PS}{6}$
$PS = 6 \times \sin 63° 1'$
$= 5.3468 \text{ cm}$

$PQ = 2PS$
$= 2 \times 5.3468$
$= 10.6936 \text{ cm}$

$OS = \sqrt{6^2 - 5.3468^2}$
$= 2.7224 \text{ cm}$

Therefore, area of ΔPOQ
$= \frac{1}{2} \times PQ \times OS$
$= \frac{1}{2} \times 10.6936 \times 2.7224$
$= 14.56 \text{ cm}^2$

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Hence, $\angle POQ = (2 \times 34° 51') \times \frac{\pi}{180°}$

$= 69° 42' \times \frac{3.142}{180°}$

$= 1.217$ rad

Area of sector $POQ = \frac{1}{2} r^2\theta$

$= \frac{1}{2}(3.5)^2(1.217)$

$= 7.454$ cm^2

In ΔPOQ, the semiperimeter, $s = \frac{3.5 + 3.5 + 4}{2}$

$s = 5.5$ cm

Area of $\Delta POQ = \sqrt{s(s-p)(s-q)(s-o)}$

$= \sqrt{5.5(5.5 - 3.5)(5.5 - 3.5)(5.5 - 4)}$

$= \sqrt{5.5(2)(2)(1.5)}$

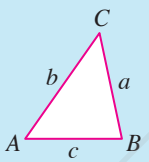
$= \sqrt{33}$

$= 5.745$ cm^2

Area of the segment $PRQ = 7.454 - 5.745$

$= 1.709$ cm^2

Recall



(a) Area of ΔABC

$= \frac{1}{2}ab \sin C$

$= \frac{1}{2}ac \sin B$

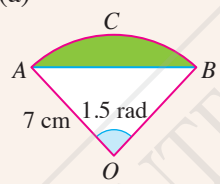
$= \frac{1}{2}bc \sin A$

(b) Formula to find area of triangle by using Heron's formula:
Area of ΔABC
$= \sqrt{s(s-a)(s-b)(s-c)}$,
where $s = \frac{a+b+c}{2}$ is the semiperimeter.

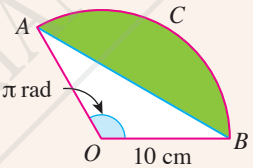
Self-Exercise 1.6

1. For each of the following sectors AOB with centre O, find the area of the segment ACB. [Use $\pi = 3.142$]

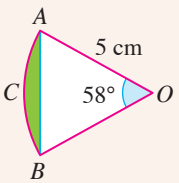
(a)



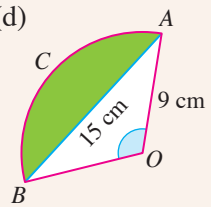
(b)



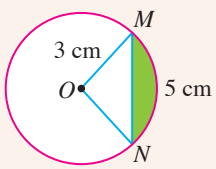
(c)



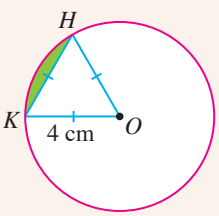
(d)



2. The diagram on the right shows sector MON of a circle with centre O and a radius of 3 cm. Given that the minor arc length MN is 5 cm, find
(a) $\angle MON$, in degrees,
(b) the area of the shaded segment, in cm^2.



3. The diagram on the right shows sector HOK of a circle with centre O and a radius of 4 cm. The length of chord HK is the same as the length of the radius of the circle. Calculate
(a) $\angle HOK$, in radians,
(b) the area of the shaded segment, in cm^2.



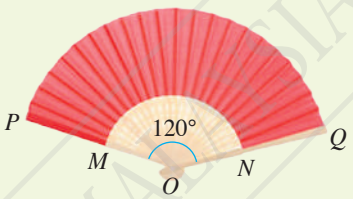
KEMENTERIAN PENDIDIKAN MALAYSIA

Solving problems involving areas of sectors

The knowledge and skills in using the area of a sector formula, $A = \frac{1}{2}r^2\theta$ or other suitable formulae can help us to solve many daily problems involving areas of sectors.

Example 10 MATHEMATICAL APPLICATIONS

The diagram on the right shows a paper fan fully spread out. The region $PQNM$ is covered by paper. Given that $OP = 15$ cm, $OM : MP = 2 : 3$ and $\angle POQ = 120°$, calculate the area covered by the paper, in cm^2.



Solution

1 . Understanding the problem

- $PQNM$ is the region covered with paper when the paper fan is opened up completely.
- Given $OP = 15$ cm, $OM : MP = 2 : 3$ and $\angle POQ = 120°$.
- Find the area, in cm^2, of the region covered by the paper.

2 . Planning the strategy

- Find the length of OM by using the ratio $OM : MP = 2 : 3$.
- Convert 120° into radians and use the formula $A = \frac{1}{2}r^2\theta$ to find the area of the sector POQ and the area of the sector MON.
- Subtract the area of the sector MON from the area of the sector POQ to obtain the area covered by the paper.

3 . Implementing the strategy

$$OM = \frac{2}{5} \times OP$$
$$= \frac{2}{5} \times 15$$
$$= 6 \text{ cm}$$

$$\theta \text{ in radians} = 120° \times \frac{\pi}{180°}$$
$$= 120° \times \frac{3.142}{180°}$$
$$= 2.0947 \text{ rad}$$

Area of sector POQ, $A = \frac{1}{2}r^2\theta$

$$A = \frac{1}{2}(15)^2(2.0947)$$
$$A = 235.65 \text{ cm}^2$$

Area of sector MON, $A = \frac{1}{2}r^2\theta$

$$A = \frac{1}{2}(6)^2(2.0947)$$
$$A = 37.70 \text{ cm}^2$$

Thus, the area covered by the paper
$= 235.65 - 37.70$
$= 197.95 \text{ cm}^2$

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4 . Check and reflect

Area of sector POQ, $A = \frac{120°}{360°} \times 3.142 \times 15^2$

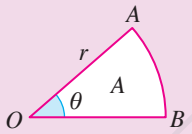
$A = 235.65 \text{ cm}^2$

Area of sector MON, $A = \frac{120°}{360°} \times 3.142 \times 6^2$

$A = 37.70 \text{ cm}^2$

Thus, the area covered by the paper
$= 235.65 - 37.70$
$= 197.95 \text{ cm}^2$

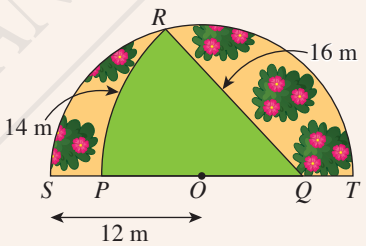
Excellent Tip



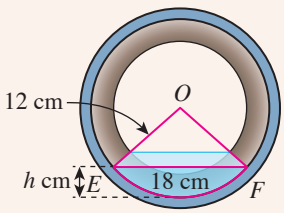
If the angle θ is in degrees, then the area of the sector of a circle, $A = \frac{\theta}{360°} \times \pi r^2$.

Self-Exercise 1.7

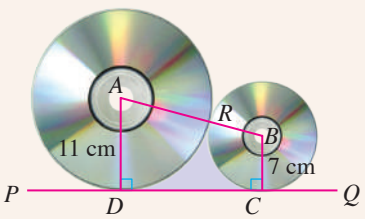
1. The diagram on the right shows a semicircular garden SRT with centre O and a radius of 12 m. The region PQR covered by grass is a sector of circle with centre Q and radius 16 m. The light brown coloured patch is fenced and planted with flowers. Given that the arc length PR is 14 m, find
(a) the length of the fence, in m, used to fence around the flowers,
(b) the area, in m^2, planted with flowers.



2. The diagram on the right shows the cross-section of a water pipe with the internal radius of 12 cm. Water flows through it to a height of h cm and the horizontal width of the water, EF is 18 cm. Calculate
(a) the value of h,
(b) the cross-section area covered by water, in cm^2.



3. The diagram on the right shows two discs with radii 11 cm and 7 cm touching each other at R. The discs are on a straight line $PDCQ$.
(a) Calculate $\angle BAD$, in degrees.
(b) Subsequently, find the shaded area, in cm^2.



4. The diagram on the right shows a wall clock showing the time 10:10 in the morning. Given that the minute hand is 8 cm, find
(a) the area swept through by the minute hand when the time shown is 10:30 in the morning, in cm^2,
(b) the angle, in radians, if the area swept through by the minute hand is 80 cm^2.

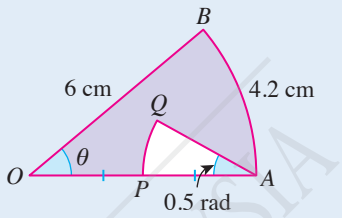


KEMENTERIAN PENDIDIKAN MALAYSIA

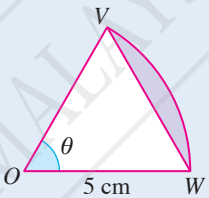
Formative Exercise 1.3

Quiz bit.ly/2NdT3uH

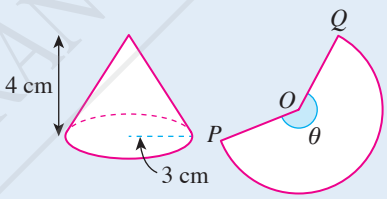
1. The diagram on the right shows sector AOB with centre O and another sector PAQ with centre A. It is given that $OB = 6$ cm, $OP = AP$, $\angle PAQ = 0.5$ rad and the arc length AB is 4.2 cm. Calculate
(a) the value of θ, in radians,
(b) the area of the shaded region, in cm^2.



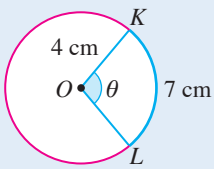
2. The diagram on the right shows sector VOW with centre O and a radius of 5 cm. Given that $OW = OV = VW$, find
(a) the value of θ, in radians,
(b) the area of the shaded segment VW, in cm^2.



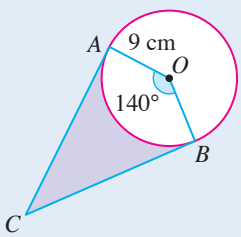
3. A cone has a base with a radius of 3 cm and a height of 4 cm. When it is opened up, it forms sector POQ as shown on the right. Given that $\angle POQ = \theta$ radian, find
(a) the value of θ,
(b) the area of sector POQ, in cm^2.



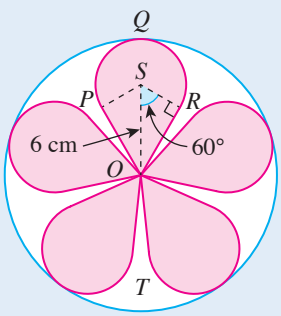
4. The diagram on the right shows a circle with centre O and a radius of 4 cm. It is given that the minor arc length KL is 7 cm.
(a) State the value of θ, in radians.
(b) Find the area of the major sector KOL, in cm^2.



5. In the diagram on the right, O is the centre of the circle with radius 9 cm. The minor arc AB subtends an angle of 140° at the centre O and the tangents at A and B meet at C. Calculate
(a) AC, in cm,
(b) the area of the kite shaped $OACB$, in cm^2,
(c) the area of the minor sector OAB, in cm^2,
(d) the area of the shaded region, in cm^2.



6. The diagram on the right shows a circular ventilation window in a hall. PQR is a major arc of a circle with centre S. The lines OP and OR are tangents to that circle. The other four panels are identical in size to $OPQR$. O is the centre of ventilation window that touches the arc PQR at Q. It is given that $OS = 6$ cm and $\angle OSR = 60°$.
(a) Show that $RS = 3$ cm.
(b) Calculate the area of the panel $OPQR$, in cm^2.
(c) The window has a rotational symmetry at O to the nth order; find the value of n and the area labelled T between two panels, in cm^2.



KEMENTERIAN PENDIDIKAN MALAYSIA

1.4 Application of Circular Measures

Study the following two situations in daily lives.



A rainbow is an optical phenomenon which displays a spectrum of colours in a circular arc. A rainbow appears when the sunlight hits the water droplets and it usually appears after a rainfall. The rainbow shown in the photo is an arc of a circle. With the formula that you have learned and the help of the latest technology, can you determine the length of this arc?

The cross-section of a train tunnel is usually in the form of a major arc of a circle. How do we find the arc length and the area of this cross-section tunnel?



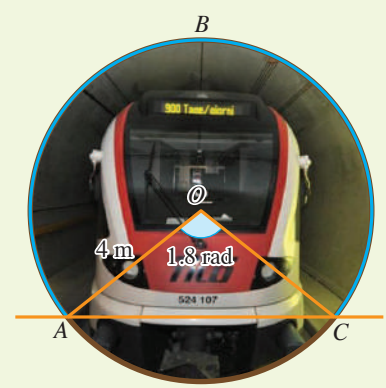
The ability to apply the formulae from circular measures, that is, the arc length, $s = r\theta$ and the area of a sector, $A = \frac{1}{2}r^2\theta$, where θ is the angle in radians and other related formulae, can help to solve the problems mentioned above.

Solving problems involving circular measures

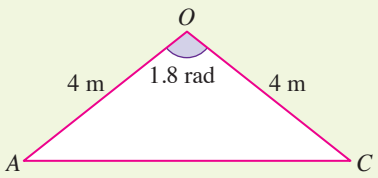
The following example shows how the formula in circular measures and other related formulae are used to solve problems related to the cross-section of a train tunnel in the form of a major segment of a circle.

Example 11

The diagram on the right shows a major segment ABC of a circular train tunnel with centre O, radius of 4 m and $\angle AOC = 1.8$ rad.
[Use $\pi = 3.142$]
(a) Show that AC is 6.266 m.
(b) Find the length of major arc ABC, in m.
(c) Find the area of the cross-section of the train tunnel, in m^2.



Solution



(a) 1.8 rad $= 1.8 \times \frac{180°}{3.142}$

$= 103° 7'$

By using the cosine rule,

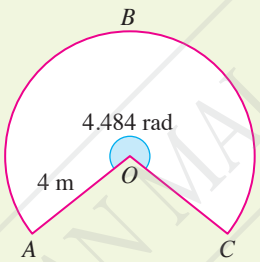
$AC^2 = OA^2 + OC^2 - 2(OA)(OC) \cos \angle AOC$

$= 4^2 + 4^2 - 2(4)(4) \cos 103° 7'$

$AC = \sqrt{4^2 + 4^2 - 2(4)(4) \cos 103° 7'}$

$= \sqrt{39.2619}$

$= 6.266$ m



(b) Reflex angle $AOC = 2\pi - 1.8$

$= 4.484$ rad

Length of major arc $ABC = r\theta$

$= 4 \times 4.484$

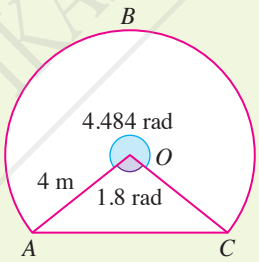
$= 17.94$ m

(c) By using the area of a triangle formula:

Area $\Delta AOC = \frac{1}{2} \times OA \times OC \times \sin \angle AOC$

$= \frac{1}{2} \times 4 \times 4 \times \sin 103° 7'$

$= 7.791$ m^2



Area of the major sector $ABC = \frac{1}{2} r^2\theta$

$= \frac{1}{2} \times 4^2 \times 4.484$

$= 35.87$ m^2

Thus, the cross-section area of the train tunnel is $7.791 + 35.87 = 43.66$ m^2

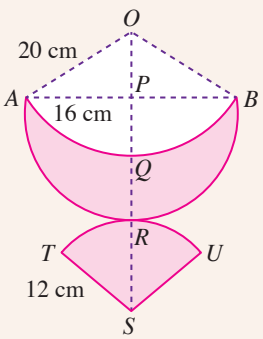
Self-Exercise 1.8

1. The diagram on the right shows a moon-shaped kite whose line of symmetry is OS. AQB is an arc of a sector from a circle with centre O and a radius of 20 cm. $APBR$ is a semicircle with centre P and a radius of 16 cm. TRU is also an arc from a circle with centre S and a radius of 12 cm. Given that the arc length of TRU is 21 cm, calculate

(a) $\angle AOB$ and $\angle TSU$, in radians,

(b) the perimeter of the kite, in cm,

(c) the area of the kite, in cm^2.



2. In the diagram on the right are three identical 20 cent coins with the same radii and touching each other. If the blue coloured region has an area of 12.842 mm^2, find the radius of each coin, in mm.



Formative Exercise 1.4

Quiz bit.ly/2FzIlu7

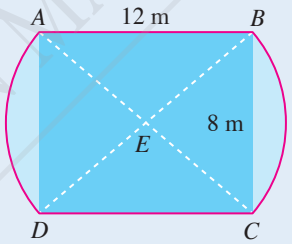
1. A cylindrical cake has a radius and a height of 11 cm and 8 cm respectively. The diagram on the right shows a uniform cross-section of a slice of a cake in the form of a sector POQ being cut out from the cylindrical cake with centre O and a radius of 11 cm. It is given that $\angle POQ = 40°$.



(a) Calculate
  (i) the perimeter of sector POQ, in cm,
  (ii) the area of sector POQ, in cm^2,
  (iii) the volume of the piece of cake that has been cut out, in cm^3.

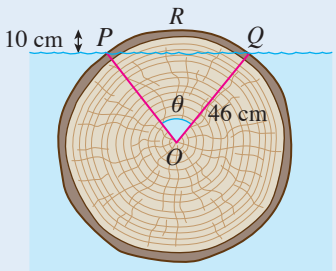
(b) If the mass of a slice of the cake that has been cut out is 150 g, calculate the mass of the whole cake, in grams.

2. The diagram on the right shows the plan of a swimming pool with a uniform depth of 1.5 m. $ABCD$ is a rectangle with the length of 12 m and the width of 8 m. AED and BEC are two sectors from a circle with centre E. Calculate



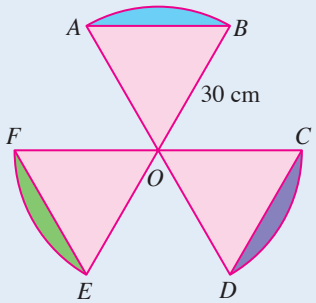
(a) the perimeter of the floor of the swimming pool, in m,
(b) the area of the floor of the swimming pool, in m^2,
(c) the volume of the water needed to fill the swimming pool, in m^3.

3. The diagram on the right shows the cross-section area of a tree trunk with a radius of 46 cm floating on the water. The points P and Q lie on the surface of the water while the highest point R is 10 cm above the surface of the water. Calculate



(a) the value of θ, in radians,
(b) the arc length PRQ, in cm,
(c) the cross-section area that is above the water, in cm^2.

4. The diagram on the right shows the logo of an ice cream company. The logo is made up of three identical sectors AOB, COD and EOF from a circle with centre O and a radius of 30 cm. It is given that $\angle AOB = \angle COD = \angle EOF = 60°$.



(a) Calculate
  (i) the arc length of AB, in cm,
  (ii) the area of sector COD, in cm^2,
  (iii) the perimeter of segment EF, in cm,
  (iv) the area of segment EF, in cm^2.

(b) The logo is casted in cement. If the thickness is uniform and is 5 cm, find the amount of cement needed, in cm^3, to make the logo.

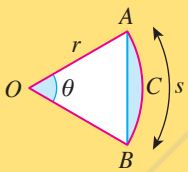
(c) If the cost of cement is RM0.50 per cm^3, find the total cost, in RM, to make the logo.

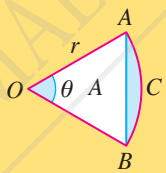
KEMENTERIAN PENDIDIKAN MALAYSIA

CHAPTER 1

REFLECTION CORNER

CIRCULAR MEASURE

- Convert radians into degrees and vice versa
  - Radians $\xrightarrow{\times \frac{180^\circ}{\pi}}$ Degrees
  - Degrees $\xrightarrow{\times \frac{\pi}{180^\circ}}$ Radians
- Arc length of a circle
  - 

  - Arc length, $s = r\theta$
  - Perimeter of segment ABC $= s + AB$
- Area of a sector of a circle
  - 

  - Area of sector, $A = \frac{1}{2}r^2\theta$
  - Area of segment ABC $= A$ – area of ΔAOB
- Applications

Journal Writing

1. Are you more inclined to measure an angle of a circle in degrees or radians? Give justification and rationale for your answers.

2. Visit the website to obtain the radius, in m, for the following six Ferris wheels:

(a) Eye on Malaysia (b) Wiener Riesenrad, Vienna (c) The London Eye
(d) Tianjin Eye, China (e) High Roller, Las Vegas (f) The Singapore Flyer

If the coordinates of the centre of each Ferris wheel is (0, 0), determine

(i) the circumference of each Ferris wheel, in m,
(ii) the area, in m^2, covered by each Ferris wheel in one complete oscillation,
(iii) the equation for each Ferris wheel.

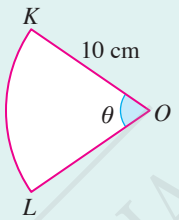
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Summative Exercise

1. The diagram on the right shows sector KOL from a circle with centre O and a radius of 10 cm. Given that the area of the sector is 60 cm^2, calculate PL 2

(a) the value of θ, in radians,

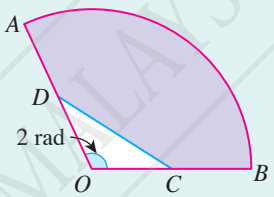
(b) the perimeter of sector KOL, in cm.



2. The diagram on the right shows sector AOB from a circle with centre O. Given that $AD = DO = OC = CB = 3$ cm, find PL 2

(a) the perimeter of the shaded region, in cm,

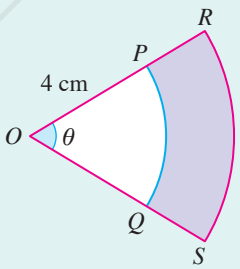
(b) the area of the shaded region, in cm^2.



3. The diagram on the right shows sectors POQ and ROS with the same centre O. Given that $OP = 4$ cm, the ratio $OP : OR = 2 : 3$ and the area of the shaded region is 10.8 cm^2, find PL 3

(a) the value of θ, in radians,

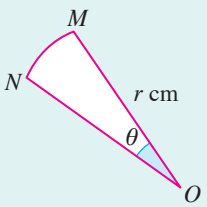
(b) the perimeter of the shaded region, in cm.



4. The diagram on the right shows sector MON from a circle with an angle of θ radian and a radius of r cm. It is given that the perimeter of the sector is 18 cm and its area is 8 cm^2. PL 3

(a) Form a pair of simultaneous equations containing r and θ.

(b) Subsequently, find the values of r and θ.

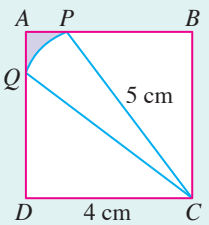


5. The diagram on the right shows a square $ABCD$ with a side of 4 cm. PQ is an arc from a circle with centre C whose radius is 5 cm. Find PL 3

(a) $\angle PCQ$, in degrees,

(b) the perimeter of the shaded region APQ, in cm,

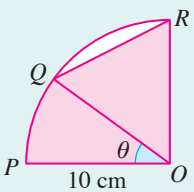
(c) the area of the shaded region APQ, in cm^2.



6. The diagram on the right shows a quadrant with centre O and a radius of 10 cm. Q is on the arc of the quadrant such that the arc lengths PQ and QR are in the ratio 2 : 3. Given that $\angle POQ = \theta$ radian, find PL 3

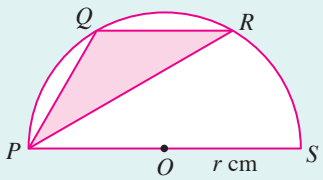
(a) the value of θ,

(b) the area of the shaded region, in cm^2.

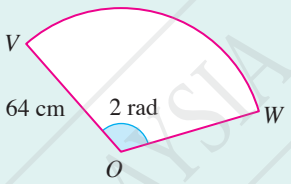


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7. In the diagram on the right, $PQRS$ is a semicircle with centre O and a radius of r cm. Given that the arc lengths of PQ, QR and RS are the same, calculate the area of the shaded region, in cm^2. Give the answer in terms of r. [Use $\pi = 3.142$] PL 5

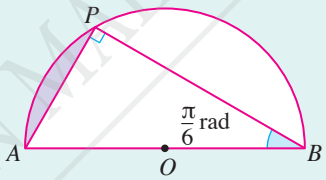


8. The diagram on the right shows a sector VOW from a circle with centre O. The arc VW subtends an angle of 2 radians at centre O. The sector is folded to make a cone such that the arc length VW is the circumference of the base of the cone. Find the height of the cone, in cm. PL 5



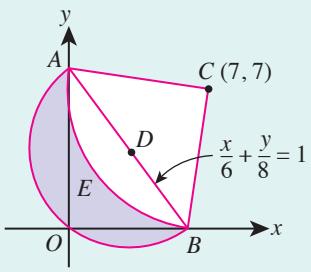
9. The diagram on the right shows semicircle $AOBP$ with O as its centre and ΔAPB is a right-angled triangle at P. Given that $AB = 16$ cm and $\angle ABP = \frac{\pi}{6}$ radian, find PL 3

(a) the length of AP, in cm,
(b) the area of ΔABP, in cm^2,
(c) the area of the shaded region, in cm^2.



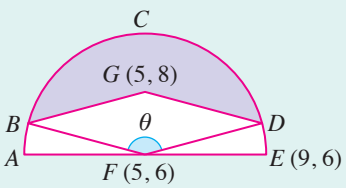
10. In the diagram on the right, AOB is a semicircle with centre D and AEB is an arc of a circle with centre $C(7, 7)$. The equation of AB is $\frac{x}{6} + \frac{y}{8} = 1$. Calculate PL 4

(a) the area of ΔABC,
(b) $\angle ACB$, in degrees,
(c) the area of the shaded region, in units^2.



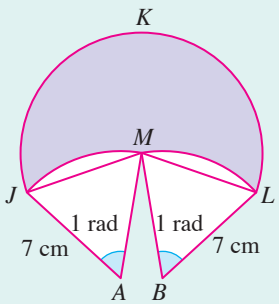
11. The diagram on the right shows a semicircle $ABCDE$ with centre F and $BGDF$ is a rhombus. It is given that the coordinates of E, F and G are (9, 6), (5, 6) and (5, 8) respectively and $\angle BFD = \theta$ radian. Calculate PL 5

(a) the value of θ, in radians,
(b) the area of sector BFD, in units^2,
(c) the area of the shaded region, in units^2.



12. The diagram on the right shows the sector of a circle $JKLM$ with centre M, and two other sectors, JAM and MBL with centres A and B respectively. Given that the major angle JML is 3.8 radians, find PL 4

(a) the radius of the sector of a circle $JKLM$, in cm,
(b) the perimeter of the shaded region, in cm,
(c) the area of sector JAM, in cm^2,
(d) the area of the shaded region, in cm^2.

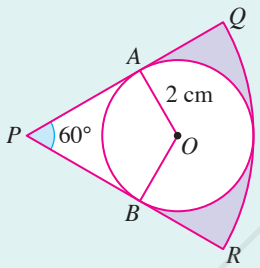


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13. The diagram on the right shows a circle with centre O and a radius of 2 cm inscribed in sector PQR from a circle with centre P. The lines PQ and PR are tangents to the circle at point A and point B. Calculate PL 4

(a) the arc length of QR, in cm,

(b) the area of the shaded region, in cm^2.

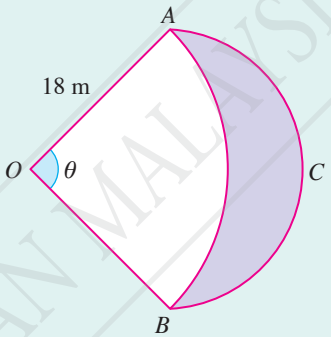


14. The diagram on the right shows the plan for a garden. AOB is a sector of a circle with centre O and a radius of 18 m and ACB is a semicircle with AB as its diameter. The sector AOB of the garden is covered with grass while creepers are planted in the shaded region ACB. Given that the area covered by grass is 243 m^2, calculate PL 4

(a) the value of θ, in radians,

(b) the length of the fence needed to enclose the creepers, in m,

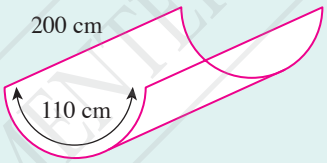
(c) the area planted with creepers, in m^2.

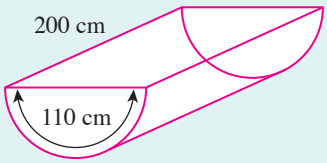


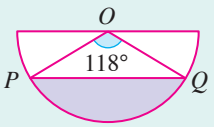
15. Hilal ties four tins of drinks together by a string as shown in the diagram. The radius of each tin is 5.5 cm. Calculate the length of the string used by Hilal, in cm. PL 5



16. A rectangular piece of aluminium measuring 200 cm by 110 cm is bent into a semicylinder as shown in the diagram. Two semicircles are used to seal up the two ends of the semicylinder so that it becomes a container to hold water as shown below. PL 5







The container is held horizontally and water is poured into the container. PQ represents the level of water in the container and O is the centre of the semicircle and $\angle POQ = 118°$.

(a) Show that the radius of the cylinder is about 35 cm, correct to the nearest cm.

(b) Calculate

(i) the area of sector POQ, in cm^2,

(ii) the area of the shaded segment, in cm^2,

(iii) the volume of water in the container, in litres.

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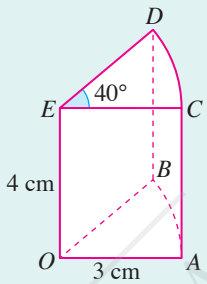
17. The diagram on the right shows a uniform prism where its cross-section is a sector of a circle with radius 3 cm. AOB and CED are identical cross-sections of the prism with points A, B, C and D lying on the curved surface of the prism. Given that the height of the prism is 4 cm and $\angle CED = 40°$, find PL 4

(a) the arc length AB, in cm,

(b) the area of sector AOB, in cm^2,

(c) the volume of the prism, in cm^3,

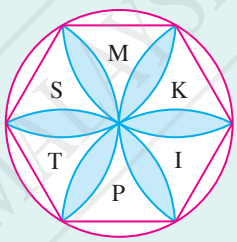
(d) the total surface area of the prism, in cm^2.



18. The mathematics society of SMK Taman Pagoh Indah organised a logo design competition for the society. The diagram on the right shows a circular logo designed by Wong made up of identical sectors from circles with radii 5 cm. Find PL 4

(a) the perimeter of the coloured region of the logo, in cm,

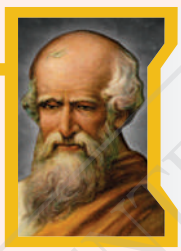
(b) the area of the coloured region of the logo, in cm^2.



MATHEMATICAL EXPLORATION

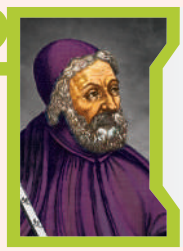
Mathematicians in the olden days suggested that the constant π is the ratio of the circumference of a circle to its diameter.
The information below shows the estimated value of π based on the opinion of four well-known mathematicians.

1



A Greek mathematician, Archimedes was able to prove that $3\frac{10}{71} < \pi < 3\frac{1}{7}$.

2



Ptolemy, a Greco-Roman mathematician showed that the estimated value of π is 3.1416.

3



Euler, a Swiss mathematician wrote that $\frac{\pi^2}{6} = 1 + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

4



Lambert, a German mathematician proved that π is an irrational number.

In our modern age, computers can evaluate the value of π to ten million digits.
Use the dynamic Desmos geometry software to explore the value of π.

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CHAPTER 2

DIFFERENTIATION



What will be learnt?

- Limit and its Relation to Differentiation
- The First Derivative
- The Second Derivative
- Application of Differentiation

List of Learning Standards

bit.ly/2NbFD2i

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Bacteria can cause various dangerous sicknesses which can be life-threatening. Bacteria produce toxins that can spoil food. Bacteria-contaminated food can cause food poisoning when consumed by humans and can be fatal if not treated immediately. Among sicknesses caused by bacteria related sicknesses are typhoid, fever and pneumonia to name a few. Do you know that the formula to calculate the number of bacteria growth of bacteria p with initial population of 1 500 is $p = 1\,500\left(\frac{1 + 5t}{t^2 + 30}\right)$, where t represents time in hours? Can you determine the growth rate of the bacteria population after 3 hours? This problem can be solved using the concept of differentiation, which is part of the field of calculus.

Video on development of bacteria colonies

bit.ly/36FWPEU

Info Corner

Isaac Newton (1643-1727 AD) and Gottfried Von Leibniz (1646-1716 AD) were two mathematicians who pioneered the study of basic principles of calculus which involved differentiation and integration.

Calculus is derived from Latin, which means a pebble used to calculate and solve a mathematical problem in ancient times.

For more info:

bit.ly/2FxmROC

Significance of the Chapter

- For a moving LRT (Light Rapid Transit), the rate of change of displacement shows its instantaneous velocity at that moment, while the rate of change of velocity shows its instantaneous acceleration.
- The concept of differentiation can be used to determine the rate of blood flow in the arteries at a particular time and can also be used to determine the rate of tumour growth or shrinkage in the human body.

Keywords

Limit	*Had*
First derivative	*Terbitan pertama*
Gradient of tangent	*Kecerunan tangen*
Second derivative	*Terbitan kedua*
Equation of tangent	*Persamaan tangen*
Equation of normal	*Persamaan normal*
Turning point	*Titik pusingan*
Rate of change	*Kadar perubahan*
Approximation	*Penghampiran*
Stationary point	*Titik pegun*
Point of inflection	*Titik lengkok balas*

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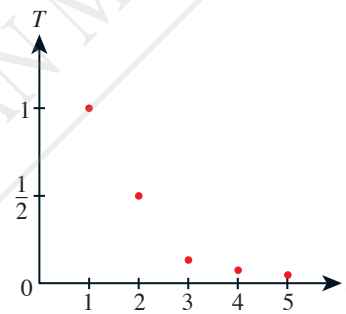
2.1 Limit and Its Relation to Differentiation

The concept of limits has been regarded as a basic concept in differential operations, just like the concept of velocity, v of an object at a certain time t is regarded as its instantaneous velocity at that moment. For example, while driving, the reading on the speedometer of a car shows a speed of 80 kmh^{-1}.

How do we get the reading of velocity 80 kmh^{-1} on the speedometer? How can we obtain the value of 80 kmh^{-1}? Using limits, we can determine this value by approximation.

The value of limit of a function when its variable approaches zero

Consider the sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ where the nth term is $T_n = \frac{1}{n}$, $n = 1, 2, 3, \ldots$



Notice the graph for this sequence as shown on the right. What will happen to the nth term as n increases indefinitely? Will the value of the nth term approach zero and yet is not zero? Can you determine the limit of that sequence?

Conduct the following discovery activity to explore the limit value of a function as its variable approaches zero.

Discovery Activity 1 Group

Aim: To explore the limit of a function when its variable approaches zero

Steps:

1. Consider the function $f(x) = \frac{x^2 + 3x}{x}$, whose domain is a set of all real numbers, $\mathbb{R}$ except zero.
2. Determine the value of $f(0)$. Are you able to get its value? Explain.
3. Copy and complete the table below for the function $f(x) = \frac{x^2 + 3x}{x}$ as x approaches zero from the left and from the right. Subsequently, sketch the graph $y = f(x)$ and determine the value of $\lim\limits_{x \to 0} \frac{x^2 + 3x}{x}$.

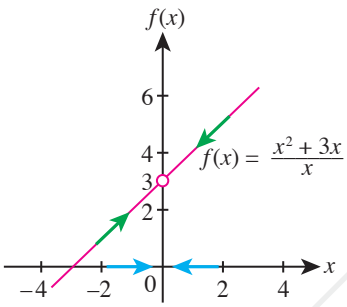
x	−0.1	−0.01	−0.001	−0.0001	...	0.0001	0.001	0.01	0.1
$f(x)$									

4. What can you conclude from the result obtained in step 2 above for the value $f(0)$ and also from the value $\lim\limits_{x \to 0} \frac{x^2 + 3x}{x}$ obtained from step 3? Discuss.

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From Discovery Activity 1, it is shown that the value of $f(0)$ cannot be determined when it is in the indeterminate form, that is, $\frac{0}{0}$. Since the limit cannot be determined by direct substitution, the value of $\lim\limits_{x \to 0} \frac{x^2 + 3x}{x}$ can be obtained as shown in the following table and diagram.

x	$f(x)$
−0.1	2.9
−0.01	2.99
−0.001	2.999
−0.0001	2.9999
0 ← ⋮	⋮ → 3
0.0001	3.0001
0.001	3.001
0.01	3.01
0.1	3.1



Technology Click

With a graphic calculator, draw the graph for the function $f(x) = \frac{x^2 + 3x}{x}$ and estimate the value of $\lim\limits_{x \to 0} f(x)$. Can the function f be defined at $x = 0$?

Discuss the effect on the limit as x approaches zero.

Based on the table above, when x approaches zero either from the left or from the right, the value of $f(x)$ approaches 3. Hence, when x approaches zero from any side, the function $f(x) = \frac{x^2 + 3x}{x}$ approaches 3, that is, when $x \to 0$, $\frac{x^2 + 3x}{x} \to 3$. The value 3 is the limit for $\frac{x^2 + 3x}{x}$ when x approaches zero and these statements can be summarised by using the notation:

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x^2 + 3x}{x} = 3$$

In general,

When x approaches a, where $x \neq a$, the limit for $f(x)$ is L can be written as $\lim\limits_{x \to a} f(x) = L$.

The steps to determine $\lim\limits_{x \to a} f(x)$, where $a \in \mathbb{R}$ are as follows:

To find the limit value of a function $f(x)$, we substitute $x = a$ directly into the function $f(x)$. If,

$f(a) \neq \frac{0}{0}$

The value of $\lim\limits_{x \to a} f(x)$ can be obtained, that is, $\lim\limits_{x \to a} f(x) = f(a)$.

$f(a) = \frac{0}{0}$

Determine $\lim\limits_{x \to a} f(x)$ by using the following methods:
- Factorisation
- Rationalising the numerator or denominator of the function.

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Example 1

Determine the limit value for each of the following functions.

(a) $\lim\limits_{x \to 4} \dfrac{3 - \sqrt{x}}{x + 2}$ (b) $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ (c) $\lim\limits_{x \to 0} \dfrac{\sqrt{x + 1} - 1}{x}$

Solution

(a) Use direct substitution.

$$\lim_{x \to 4} \frac{3 - \sqrt{x}}{x + 2} = \frac{3 - \sqrt{4}}{4 + 2} = \frac{3 - 2}{4 + 2} = \frac{1}{6}$$

(b) When $x = 1$, $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ is in the indeterminate form, $\dfrac{0}{0}$.

Thus, we need to factorise and eliminate the common factor before we can use direct substitution.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$$
$$= \lim_{x \to 1} \frac{(x + 1)(x - 1)}{x - 1}$$ ← Factorise the numerator and then eliminate the common factor
$$= \lim_{x \to 1} (x + 1)$$
$$= 1 + 1$$ ← Direct substitution
$$= 2$$

Technology Click

Sketch a graph for each of the following functions.

(a) $f(x) = \dfrac{x^2 - 1}{x - 1}, x \neq 1$

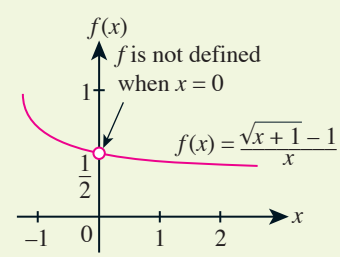
(b) $f(x) = x + 1$

From the graph, find the limit for each function as x approaches 1.

With the help of dynamic geometry software, draw a graph of each function. Can the software differentiate between the two graphs? Explain.

(c) When using direct substitution, the indeterminate form, $\dfrac{0}{0}$ will be obtained. Therefore, there is a need to rationalise the numerator by multiplying it with its conjugate, which is $\sqrt{x + 1} + 1$.

$$\lim_{x \to 0} \frac{\sqrt{x + 1} - 1}{x}$$
$$= \lim_{x \to 0} \left[\left(\frac{\sqrt{x + 1} - 1}{x}\right)\left(\frac{\sqrt{x + 1} + 1}{\sqrt{x + 1} + 1}\right)\right]$$ ← Multiply the numerator with its conjugate
$$= \lim_{x \to 0} \frac{(x + 1) - 1}{x\left(\sqrt{x + 1} + 1\right)}$$ ← $(a - b)(a + b) = a^2 - b^2$
$$= \lim_{x \to 0} \frac{x}{x\left(\sqrt{x + 1} + 1\right)}$$ ← Eliminate the common factor
$$= \lim_{x \to 0} \frac{1}{\sqrt{x + 1} + 1}$$
$$= \frac{1}{\sqrt{0 + 1} + 1}$$ ← Direct substitution
$$= \frac{1}{1 + 1}$$
$$= \frac{1}{2}$$

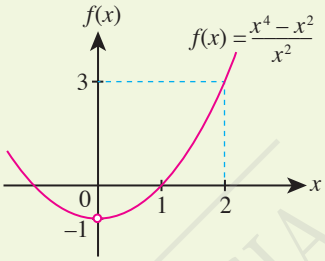


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Example 2

The diagram on the right shows a part of the graph $f(x) = \dfrac{x^4 - x^2}{x^2}$, $x \neq 0$. Based on the graph, find

(a) $f(0)$ (b) $\lim\limits_{x \to 0} f(x)$ (c) $\lim\limits_{x \to 2} f(x)$



Solution

(a) There is no value for $x = 0$. Therefore, $f(0)$ cannot be defined at $x = 0$.

(b) When $x \to 0$ either from the left or from the right, $f(x) \to -1$. Thus, $\lim\limits_{x \to 0} f(x) = -1$.

(c) When $x \to 2$ either from the left or from the right, $f(x) \to 3$. Thus, $\lim\limits_{x \to 2} f(x) = 3$.

Self-Exercise 2.1

1. Find the limit for each of the following functions when $x \to 0$.

(a) $x^2 + x - 3$ (b) $\sqrt{x + 1}$ (c) $\dfrac{x + 4}{x - 2}$ (d) $\dfrac{a}{ax + a}$

2. Determine the limit for each of the following functions.

(a) $\lim\limits_{x \to 0} (3x - 1)$ (b) $\lim\limits_{x \to -3} \sqrt{10 - 2x}$ (c) $\lim\limits_{x \to -3} \dfrac{x^2 + x - 6}{x + 3}$

(d) $\lim\limits_{x \to 6} \dfrac{x - 6}{x^2 - 36}$ (e) $\lim\limits_{x \to 2} \dfrac{x^2 - 3x + 2}{x^2 - 4}$ (f) $\lim\limits_{x \to 0} \dfrac{1 - \sqrt{2x + 1}}{2x^2 - x}$

(g) $\lim\limits_{x \to 4} \dfrac{x - 4}{\sqrt{x} - 2}$ (h) $\lim\limits_{x \to 3} \dfrac{3 - \sqrt{2x + 3}}{x - 3}$ (i) $\lim\limits_{x \to -2} \dfrac{x + 2}{\sqrt{5x + 14} - 2}$

3. Find the value for each of the following limits.

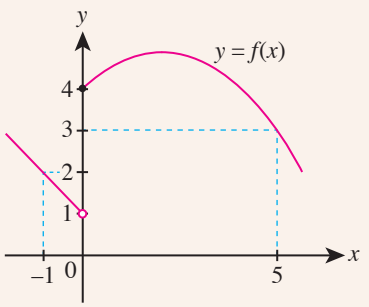
(a) $\lim\limits_{x \to 0} \dfrac{x^2 - 2x}{x^3 - 4x}$ (b) $\lim\limits_{x \to 3} \dfrac{x^2 - 4x + 3}{2x^2 - 5x - 3}$ (c) $\lim\limits_{x \to 3} \dfrac{x^3 - 5x^2 + 6x}{x^2 - 3x}$

(d) $\lim\limits_{x \to 0} \dfrac{5x}{3 - \sqrt{x + 9}}$ (e) $\lim\limits_{x \to 4} \dfrac{x - 4}{2 - \sqrt{8 - x}}$ (f) $\lim\limits_{x \to 7} \dfrac{\sqrt{x + 2} - 3}{x - 7}$

4. The diagram on the right shows a part of the function graph $y = f(x)$.

(a) Based on the graph,
  (i) find $f(0)$,
  (ii) determine whether $\lim\limits_{x \to 0} f(x)$ exists or not. Explain.

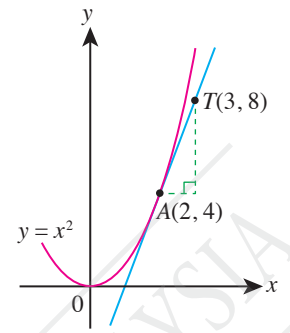
(b) Then, find
  (i) $\lim\limits_{x \to -1} f(x)$
  (ii) $\lim\limits_{x \to 5} f(x)$



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First derivative of a function $f(x)$ by using first principles

A tangent to a curve at a point is a straight line that touches the curve at only that point. In the diagram on the right, straight line AT is a tangent to the curve $y = x^2$ at the point A with the coordinates of A and T being $(2, 4)$ and $(3, 8)$ respectively.



$$\text{Gradient of tangent } AT = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 4}{3 - 2} = 4$$

What method can be used to find the gradient of the tangent to the curve $y = x^2$ at other points on the curve, such as $B(3, 9)$?

Using a graph to obtain the gradient can be difficult and also inaccurate. There are other methods to find the gradient of the curve at a particular point, that is by using the idea of limits as in the discovery activity below.

Information Corner

Gradient of the curve is also known as gradient of the tangent.

Discovery Activity 2

Group | 21st cl | STEM | CT

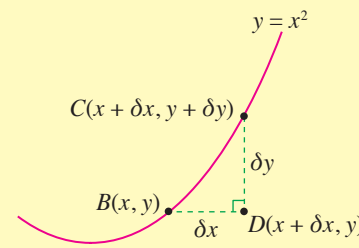
Aim: To explore the gradient of the tangent function and the gradient of the tangent to the curve $y = x^2$ at point $B(3, 9)$ using the idea of limits

ggbm.at/fwcrewdm

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Consider the curve $y = x^2$ and the line that passes through point $B(3, 9)$ and point $C(4, 16)$ on the graph.
3. The value $m = 7$ is the gradient of line BC.
4. Drag point C nearer to point B and observe the change in the value of m.
5. Record the change in value m as point C moves closer to point B.
6. Let the coordinates of $B(3, 9)$ be (x, y) and the coordinates of $C(4, 16)$ be $(x + \delta x, y + \delta y)$, where δx represents the change in the value of x, and δy represents the change in the value of y. Copy and complete the following table.

δx	$x + \delta x$	$y + \delta y$	δy	$\frac{\delta y}{\delta x}$
1	4	16	7	7
0.5	3.5	12.25	3.25	
0.05				
0.005				

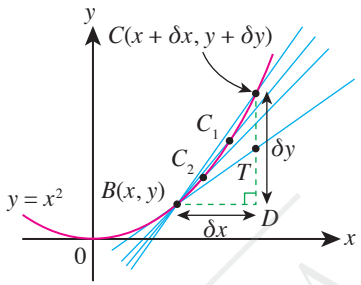


7. When δx approaches 0, what happens to the value of $\frac{\delta y}{\delta x}$? Compare this result with the result obtained in step 5.

From Discovery Activity 2, note that $B(x, y)$ and $C(x + \delta x, y + \delta y)$ are two points close to each other on the curve $y = x^2$.

Hence,

$$\text{Gradient of the line } BC = \frac{CD}{BD}$$
$$= \frac{(y + \delta y) - y}{(x + \delta x) - x}$$
$$= \frac{\delta y}{\delta x}$$



As point C approaches point B along the curve, the line BC changes and becomes BC_1 and then becomes BC_2, that is, the value of δx gets smaller and approaches zero, $\delta x \to 0$. When point C is at point B, the line becomes a tangent at B. Hence,

$$\text{Gradient of the curve at } B = \text{Gradient of tangent } BT$$
$$= \text{Value of } \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

Hence, for the curve $y = f(x)$, the gradient function of the tangent at any point can be obtained by finding $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$. $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ is called the first derivative of the function with respect to x and is written with the symbol $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \lim_{\delta x \to 0} \frac{f(x + \delta x) - f(x)}{\delta x}$$

The gradient function of a tangent $\frac{dy}{dx}$ can be used to find the gradient of the tangent at any point $(x, f(x))$ on the curve $y = f(x)$.

For example, take the earlier function $y = f(x) = x^2$.

$$\delta y = f(x + \delta x) - f(x)$$
$$= (x + \delta x)^2 - x^2$$
$$= x^2 + 2x(\delta x) + (\delta x)^2 - x^2$$
$$= 2x(\delta x) + (\delta x)^2$$
$$\frac{\delta y}{\delta x} = \frac{2x(\delta x) + (\delta x)^2}{\delta x}$$ ← Divide both sides by δx
$$= 2x + \delta x$$

Then,

$$\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$
$$= \lim_{\delta x \to 0} (2x + \delta x)$$
$$= 2x + 0$$
$$\frac{dy}{dx} = 2x$$ ← Gradient of the tangent function

HISTORY GALLERY



The concept of limit was first introduced explicitly by Sir Isaac Newton. He said that limits was the basic concept in calculus and explained that the most important limit concept is "getting smaller and smaller than the differences between any two given values".

Information Corner

- Symbol δx is read as "delta x", which represents a small change in x.
- Symbol δy is read as "delta y", which represents a small change in y.

Excellent Tip

$\frac{dy}{dx}$ does not mean dy divide by dx but $\frac{dy}{dx}$ is the symbol for $\lim \frac{\delta y}{\delta x}$ when $\delta x \to 0$.

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Hence, the gradient of the tangent to the curve $y = x^2$ at point $B(3, 9)$ is $\frac{dy}{dx} = 2x = 2(3) = 6$.

In general, the process of determining the gradient function $\frac{dy}{dx}$ or the first derivative of a function $y = f(x)$ is by using the idea of $\lim\limits_{\delta x \to 0} \frac{\delta y}{\delta x}$ which is known as **differentiation using first principles**.

Example 3

Find $\frac{dy}{dx}$ by using first principles for each of the following functions $y = f(x)$.

(a) $y = 3x$ (b) $y = 3x^2$ (c) $y = 3x^3$

Solution

(a) Given $y = f(x) = 3x$

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x)\\ &= 3(x + \delta x) - 3x\\ &= 3x + 3\delta x - 3x\\ &= 3\delta x\\ \frac{\delta y}{\delta x} &= 3\end{aligned}$$

$$\begin{aligned}\text{Hence, } \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\\ &= \lim_{\delta x \to 0} 3\\ \frac{dy}{dx} &= 3\end{aligned}$$

(b) Given $y = f(x) = 3x^2$

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x)\\ &= 3(x + \delta x)^2 - 3x^2\\ &= 3[x^2 + 2x(\delta x) + (\delta x)^2] - 3x^2\\ &= 3x^2 + 6x(\delta x) + 3(\delta x)^2 - 3x^2\\ &= 6x(\delta x) + 3(\delta x)^2\\ \frac{\delta y}{\delta x} &= 6x + 3\delta x\end{aligned}$$

$$\begin{aligned}\text{Hence, } \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\\ &= \lim_{\delta x \to 0} (6x + 3\delta x)\\ &= 6x + 3(0)\\ \frac{dy}{dx} &= 6x\end{aligned}$$

(c) Given $y = f(x) = 3x^3$

$$\begin{aligned}\delta y &= f(x + \delta x) - f(x)\\ &= 3(x + \delta x)^3 - 3x^3\\ &= 3(x + \delta x)(x + \delta x)^2 - 3x^3\\ &= 3(x + \delta x)[x^2 + 2x(\delta x) + (\delta x)^2] - 3x^3\\ &= 3[x^3 + 2x^2(\delta x) + x(\delta x)^2 + x^2(\delta x) + 2x(\delta x)^2 + (\delta x)^3] - 3x^3\\ &= 3[x^3 + 3x^2(\delta x) + 3x(\delta x)^2 + (\delta x)^3] - 3x^3\\ &= 3x^3 + 9x^2(\delta x) + 9x(\delta x)^2 + 3(\delta x)^3 - 3x^3\\ &= 9x^2(\delta x) + 9x(\delta x)^2 + 3(\delta x)^3\\ \frac{\delta y}{\delta x} &= 9x^2 + 9x(\delta x) + 3(\delta x)^2\end{aligned}$$

$$\begin{aligned}\text{Hence, } \frac{dy}{dx} &= \lim_{\delta x \to 0} \frac{\delta y}{\delta x}\\ &= \lim_{\delta x \to 0} [9x^2 + 9x(\delta x) + 3(\delta x)^2]\\ &= 9x^2 + 9x(0) + 3(0)^2\\ \frac{dy}{dx} &= 9x^2\end{aligned}$$

Excellent Tip

Steps to determine $\frac{dy}{dx}$ for any function $f(x)$ using first principles:

1. Consider two points $A(x, y)$ and $B(x + \delta x, y + \delta y)$ on the curve.
2. Find δy with $\delta y = f(x + \delta x) - f(x)$.
3. Obtain the ratio $\frac{\delta y}{\delta x}$.
4. Take the limit of $\frac{\delta y}{\delta x}$ when $\delta x \to 0$.

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Self-Exercise 2.2

1. Find $\frac{dy}{dx}$ by using first principles for each of the following functions $y = f(x)$.

(a) $y = x$ (b) $y = 5x$ (c) $y = -4x$ (d) $y = 6x^2$

(e) $y = -x^2$ (f) $y = 2x^3$ (g) $y = \frac{1}{2}x^2$ (h) $y = \frac{1}{x}$

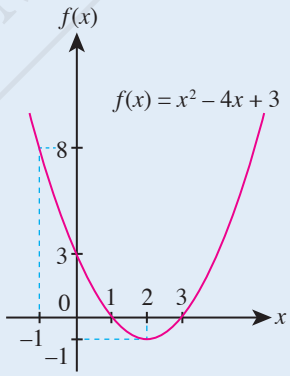
2. Given $y = 2x^2 - x + 7$, find $\frac{dy}{dx}$ by using first principles.

3. By using first principles, find the gradient function to the curve $y = 3 + x - x^2$.

Formative Exercise 2.1

Quiz bit.ly/2QEq2KN

1. The diagram on the right shows a part of the graph $f(x) = x^2 - 4x + 3$.



(a) From the graph, find each of the following.

(i) $\lim_{x \to -1} f(x)$ (ii) $\lim_{x \to 0} f(x)$ (iii) $\lim_{x \to 1} f(x)$

(iv) $\lim_{x \to 2} f(x)$ (v) $\lim_{x \to 3} f(x)$ (vi) $\lim_{x \to 4} f(x)$

(b) Find the possible values of a if $\lim_{x \to a} f(x) = 8$.

(c) (i) Determine the gradient of the tangent function, $\frac{dy}{dx}$ of the graph by using first principles.

(ii) Then, determine the gradient of the tangent at point (4, 3).

2. Find the value for each of the following limits.

(a) $\lim_{x \to 0} (x^2 - 6x + 9)$ (b) $\lim_{x \to 2} \sqrt[3]{x^4 - 2x^2}$ (c) $\lim_{x \to 9} \frac{9 - x}{x^2 - 81}$

(d) $\lim_{x \to 2} \frac{x^2 - x - 2}{x - 2}$ (e) $\lim_{x \to 1} \frac{x^3 - x}{x - 1}$ (f) $\lim_{x \to 5} \frac{x^2 - 7x + 10}{x^2 - 25}$

3. Determine the limit value for each of the following functions.

(a) $\lim_{x \to 0} \frac{\sqrt{1 + 2x} - \sqrt{1 - 2x}}{x}$ (b) $\lim_{x \to 4} \frac{3 - \sqrt{x + 5}}{x - 4}$ (c) $\lim_{x \to 3} \frac{x^2 - 5x + 6}{2 - \sqrt{x + 1}}$

4. (a) Given that $\lim_{x \to 2} \frac{x^2 - k}{3x - 6} = \frac{4}{3}$, find the value of k.

(b) If $\lim_{x \to -1} \frac{x^2 - 2x - h}{kx + 2} = -2$, find the value of $h + k$.

5. Differentiate the following functions with respect to x by using first principles.

(a) $y = 5x - 8$ (b) $y = x^2 - x$ (c) $y = (x + 1)^2$ (d) $y = \frac{1}{4x}$

6. The displacement of a squirrel running on a straight cable for t seconds is given by $s(t) = t^2 - 3t$, where $t \geqslant 0$. By using first principles, find the velocity of the squirrel when $t = 5$.

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2.2 The First Derivative

First derivative formula for the function $y = ax^n$, where a is a constant and n is an integer

Let us look at Example 3 on page 36 again. The first derivative of the function $y = 3x$, $y = 3x^2$ and $y = 3x^3$ by using first principles seems to follow a pattern as shown in the table on the right.

Function	$\frac{dy}{dx}$	Pattern
$y = 3x$	3	$3(1x^{1-1})$
$y = 3x^2$	$6x$	$3(2x^{2-1})$
$y = 3x^3$	$9x^2$	$3(3x^{3-1})$

From the given pattern for the function $y = ax^n$, where a is a constant and n is an integer, we can deduce the first derivative formula for the function as follows.

If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$ or $\frac{d}{dx}(ax^n) = anx^{n-1}$

Excellent Tip

For $y = ax^n$,
- If $n = 1$, $\frac{dy}{dx} = a$
- If $n = 0$, $\frac{dy}{dx} = 0$

Three notations used to indicate the first derivative of a function $y = ax^n$ are as follows.

1. If $y = 3x^2$, then $\frac{dy}{dx} = 6x$ — $\frac{dy}{dx}$ is read as differentiating y with respect to x.
2. If $f(x) = 3x^2$, then $f'(x) = 6x$ — $f'(x)$ is known as the gradient function for the curve $y = f(x)$ because this function can be used to find the gradient of the curve at any point on the curve.
3. $\frac{d}{dx}(3x^2) = 6x$ — If differentiating $3x^2$ with respect to x, the result is $6x$.

Determining the first derivative of an algebraic function

The following discovery activity will compare the function graph $f(x)$ and its gradient function graph, $f'(x)$ by using the dynamic Desmos geometry software.

Discovery Activity 3 Group STEM CT

Aim: To compare the function graph $f(x)$ with its gradient function graph, $f'(x)$

Steps:

bit.ly/306oAEg

1. Scan the QR code on the right or visit the link below it.
2. Pay attention to the graph $f(x) = x^2$ drawn on the plane.
3. Click the button $(a, f(a))$ to see the coordinates where the tangent touches the graph $f(x)$.
4. Then, click the button $f'(x) = \frac{d}{dx}[f(x)]$ to see the graph $f'(x)$, which is the gradient function graph for $f(x)$. Then, click the button $(a, f'(a))$ to see the coordinates on the graph $f'(x)$.

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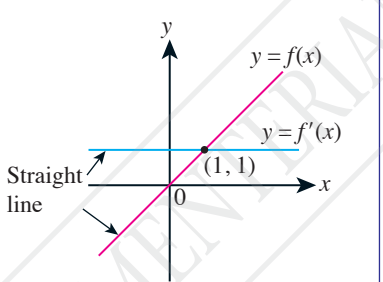
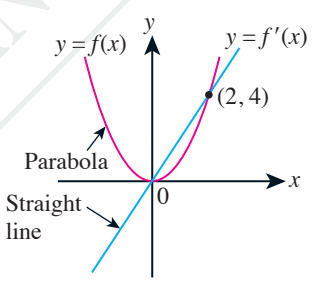
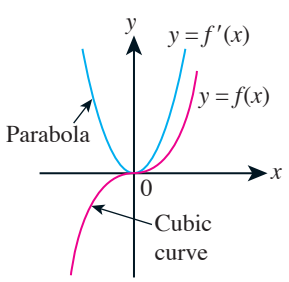
5. Drag the slider a to change the point where the tangent touches the curve $f(x)$.
6. Compare the function graph $f(x)$ with its gradient function graph, $f'(x)$. What can you deduce about the two graphs when a changes?
7. Copy and complete the table below to find the gradient of the curve $y = x^2$ at the given x-coordinates. The gradient of the curve can be obtained by locating the y-coordinate of the point on the graph $f'(x)$.

x-coordinates	−3	−2	−1	0	1	2	3
Gradient of the curve							

8. By using the first derivative formula which has been learnt earlier, determine the function $f'(x)$. Then, substitute the values of the x-coordinates from the table above into the function $f'(x)$ to verify and check the gradient of the curve obtained in step 7.
9. Continue to explore by using other functions such as cubic functions, then compare the type and shape of this function graph with its gradient function graph.
10. Make a conclusion based on your findings.

From Discovery Activity 3 results, we gather that:

- The comparison between the graph $f(x)$ and its gradient function, $f'(x)$ for each of the three polynomial functions in the form $y = f(x) = ax^n$, where $a = 1$ and the highest power of the polynomial, $n = 1, 2$ and 3, can be summarised as shown below.

Graph $y = f(x) = x$ and $y = f'(x) = 1$	Graph $y = f(x) = x^2$ and $y = f'(x) = 2x$	Graph $y = f(x) = x^3$ and $y = f'(x) = 3x^2$
		

- The steps to obtain the gradient of the curve $f(x)$ at a point are as follows.

Find the gradient function $f'(x)$ for the function $f(x) = ax^n$ by using the following formula:

If $f(x) = ax^n$, where a is a constant and n is an integer, then $f'(x) = anx^{n-1}$.

→ Substitute the value of x into the gradient function.

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The process of determining the gradient function $f'(x)$ from a function $y = f(x)$ is known as **differentiation**. The gradient function is also known as the **first derivative of the function** or **the derived function** or **differentiating coefficient of y with respect to x**.

Example 4

Differentiate each of the following with respect to x.

(a) $-\frac{2}{3}x^6$ (b) $y = \frac{1}{5}\sqrt{x}$ (c) $f(x) = \frac{3}{8x^2}$

Solution

(a) $\frac{d}{dx}\left(-\frac{2}{3}x^6\right) = -\frac{2}{3}(6x^{6-1})$

$= -\frac{2}{3}(6x^5)$

$\frac{d}{dx}\left(-\frac{2}{3}x^6\right) = -4x^5$

(b) $y = \frac{1}{5}\sqrt{x}$

$= \frac{1}{5}x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{5}\left(\frac{1}{2}x^{\frac{1}{2}-1}\right)$

$= \frac{1}{10}x^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{10\sqrt{x}}$

(c) $f(x) = \frac{3}{8x^2}$

$= \frac{3}{8}x^{-2}$

$f'(x) = \frac{3}{8}(-2x^{-2-1})$

$= -\frac{3}{4}x^{-3}$

$f'(x) = -\frac{3}{4x^3}$

Example 5

(a) If $f(x) = \frac{3}{4}x^4$, find $f'(-1)$ and $f'\left(\frac{1}{3}\right)$.

(b) Given that $y = 9\sqrt[3]{x}$, find the value of $\frac{dy}{dx}$ when $x = 8$.

Solution

(a) $f(x) = \frac{3}{4}x^4$

$f'(x) = \frac{3}{4}(4x^{4-1})$

$= 3x^3$

$f'(-1) = 3(-1)^3$

$= -3$

$f'\left(\frac{1}{3}\right) = 3\left(\frac{1}{3}\right)^3$

$= \frac{1}{9}$

(b) $y = 9\sqrt[3]{x}$

$= 9x^{\frac{1}{3}}$

$\frac{dy}{dx} = 9\left(\frac{1}{3}x^{\frac{1}{3}-1}\right)$

$= 3x^{-\frac{2}{3}}$

When $x = 8$, $\frac{dy}{dx} = 3(8)^{-\frac{2}{3}}$

$= \frac{3}{4}$

Information Corner

A gradient function of a curve is a function while the gradient of a curve at a given point has a numeric value.

For example, for the curve $y = 2x^3$, its gradient function is $\frac{dy}{dx} = 2(3x^{3-1}) = 6x^2$ and the gradient at point (1, 2) is $\frac{dy}{dx} = 6(1)^2 = 6$.

The derivative of a function which contains terms algebraically added or subtracted can be done by differentiating each term separately.

If $f(x)$ and $g(x)$ are functions, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$$

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Example 6

Differentiate each of the following with respect to x.

(a) $5x^3 + \frac{3}{4}x^4$ (b) $x\left(\sqrt{x} - 9\right)$ (c) $\frac{(2x+1)(x-1)}{x}$

Solution

(a) $\frac{d}{dx}\left(5x^3 + \frac{3}{4}x^4\right) = \frac{d}{dx}(5x^3) + \frac{d}{dx}\left(\frac{3}{4}x^4\right)$ ← Differentiate each term separately

$= 5(3x^{3-1}) + \frac{3}{4}\left(4x^{4-1}\right)$

$\frac{d}{dx}\left(5x^3 + \frac{3}{4}x^4\right) = 15x^2 + 3x^3$

(b) Let $f(x) = x\left(\sqrt{x} - 9\right)$

$= x^{\frac{3}{2}} - 9x$

$f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} - 9(1x^{1-1})$ ← Differentiate each term separately

$= \frac{3}{2}x^{\frac{1}{2}} - 9$

$f'(x) = \frac{3}{2}\sqrt{x} - 9$

(c) Let $y = \frac{(2x+1)(x-1)}{x}$

$= \frac{2x^2 - x - 1}{x}$

$= 2x - 1 - x^{-1}$

$\frac{dy}{dx} = \frac{d}{dx}(2x) - \frac{d}{dx}(1) - \frac{d}{dx}(x^{-1})$ ← Differentiate each term separately

$= 2x^{1-1} - 0x^{0-1} - (-1x^{-1-1})$

$= 2 + x^{-2}$

$\frac{dy}{dx} = 2 + \frac{1}{x^2}$

Self-Exercise 2.3

1. Find the first derivative for each of the following functions with respect to x.

(a) $\frac{4}{5}x^{10}$ (b) $-2x^4$ (c) $\frac{3}{4x^8}$ (d) $\frac{6}{\sqrt[3]{x}}$ (e) $-12\sqrt[3]{x^2}$

2. Differentiate each of the following functions with respect to x.

(a) $4x^2 + 6x - 1$ (b) $\frac{4}{5}\sqrt{x} + \frac{2}{\sqrt{x}}$ (c) $(9 - 4x)^2$

3. Differentiate each of the following functions with respect to x.

(a) $y = 4x^2\left(5 - \sqrt{x}\right)$ (b) $y = \left(x^2 + \frac{4}{x}\right)^2$ (c) $y = \frac{(4x-1)(1-x)}{\sqrt{x}}$

4. Find the value of $\frac{dy}{dx}$ for each of the given value of x.

(a) $y = x^2 - 2x, x = \frac{1}{2}$ (b) $y = \sqrt{x}(2 - x), x = 9$ (c) $y = \frac{x^2 + 4}{x^2}, x = 2$

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First derivative of composite function

To differentiate the function $y = (2x + 3)^2$, we expand the function into $y = 4x^2 + 12x + 9$ before it is differentiated term by term to get $\frac{dy}{dx} = 8x + 12$.

However, what if we want to differentiate the function $y = (2x + 3)^4$? Then $(2x + 3)^4$ will be too difficult to expand unless we consider the function as a composite function consisting of two simple functions. Let's explore the following method.

Discovery Activity 4 Individual

Aim: To explore a different method to differentiate a function which is in the form $y = (ax + b)^n$, where $a \neq 0$

Steps:

1. Consider the function $y = (2x + 3)^2$.
2. Expand the expression $(2x + 3)^2$ and determine $\frac{dy}{dx}$ by differentiating each term separately.
3. If $u = 2x + 3$,
   (a) express y as a function of u,
   (b) find $\frac{du}{dx}$ and $\frac{dy}{du}$,
   (c) determine $\frac{dy}{du} \times \frac{du}{dx}$ in terms of x and simplify your answer.
4. Compare the methods in steps 2 and 3. Are the answers the same? Which method will you choose? Give your reasons.

From Discovery Activity 4, we found that there are other methods to differentiate functions like $y = (2x + 3)^2$. However, the method used in step 3 is much easier to get the derivative of an expression which is in the form $(ax + b)^n$, where $a \neq 0$, that is difficult to expand.

For function $y = f(x) = (2x + 3)^2$:

Let $u = h(x) = 2x + 3$

Then, $y = g(u) = u^2$

In this case, y is a function of u and u is a function of x. Hence, we say that $y = f(x)$ is a composite function with $y = g(u)$ and $u = h(x)$.

To differentiate a function like this, we will introduce a simple method known as the **chain rule**, which is:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

QR Access

To prove the chain rule using the idea of limits

bit.ly/2tGmLS8

Information Corner

The expression $(2x + 3)^4$ can be expanded using the Binomial theorem.

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In general, the first derivative of a composite function is as follows:

If $y = g(u)$ and $u = h(x)$, then differentiating y with respect to x will give

$$f'(x) = g'(u) \times h'(x)$$

That is, $$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 7

Differentiate each of the following with respect to x.

(a) $y = (3x^2 - 4x)^7$ (b) $y = \dfrac{1}{(2x+3)^3}$ (c) $y = \sqrt{6x^2 + 8}$

Solution

(a) Let $u = 3x^2 - 4x$ and $y = u^7$

Then, $\dfrac{du}{dx} = 6x - 4$ and $\dfrac{dy}{du} = 7u^6$

With chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 7u^6(6x - 4)$$
$$= 7(3x^2 - 4x)^6(6x - 4)$$
$$= (42x - 28)(3x^2 - 4x)^6$$
$$\frac{dy}{dx} = 14(3x - 2)(3x^2 - 4x)^6$$

(b) Let $u = 2x + 3$ and $y = \dfrac{1}{u^3} = u^{-3}$

Then, $\dfrac{du}{dx} = 2$ and $\dfrac{dy}{du} = -3u^{-3-1} = -\dfrac{3}{u^4}$

With chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= -\frac{3}{u^4}(2)$$
$$\frac{dy}{dx} = -\frac{6}{(2x+3)^4}$$

(c) Let $u = 6x^2 + 8$ and $y = \sqrt{u} = u^{\frac{1}{2}}$

Then, $\dfrac{du}{dx} = 12x$ and $\dfrac{dy}{du} = \dfrac{1}{2}u^{\frac{1}{2}-1} = \dfrac{1}{2}u^{-\frac{1}{2}} = \dfrac{1}{2\sqrt{u}}$

With chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \frac{1}{2\sqrt{u}}(12x)$$
$$= \frac{12x}{2\sqrt{6x^2 + 8}}$$
$$\frac{dy}{dx} = \frac{6x}{\sqrt{6x^2 + 8}}$$

Information Corner

In general, for functions in the form $y = u^n$, where u is a function of x, then $\dfrac{dy}{du} = nu^{n-1}\dfrac{du}{dx}$ or $\dfrac{d}{dx}(u^n) = nu^{n-1}\dfrac{du}{dx}$.
This formula can be used to differentiate directly for Example 7.

Self-Exercise 2.4

1. Differentiate each of the following expressions with respect to x.

(a) $(x + 4)^5$ (b) $(2x - 3)^4$ (c) $\dfrac{1}{3}(6 - 3x)^6$ (d) $(4x^2 - 5)^7$

(e) $\left(\dfrac{1}{6}x + 2\right)^8$ (f) $\dfrac{2}{3}(5 - 2x)^9$ (g) $(1 - x - x^2)^3$ (h) $(2x^3 - 4x + 1)^{-10}$

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2. Differentiate each of the following expressions with respect to x.

(a) $\frac{1}{3x+2}$ (b) $\frac{1}{(2x-7)^3}$ (c) $\frac{5}{(3-4x)^5}$ (d) $\frac{3}{4(5x-6)^8}$

(e) $\sqrt{2x-7}$ (f) $\sqrt{6-3x}$ (g) $\sqrt{3x^2+5}$ (h) $\sqrt{x^2-x+1}$

3. Find the value of $\frac{dy}{dx}$ for each of the given value of x or y.

(a) $y = (2x+5)^4$, $x = 1$ (b) $y = \sqrt{5-2x}$, $x = \frac{1}{2}$ (c) $y = \frac{1}{2x-3}$, $y = 1$

First derivative of a function involving product and quotient of algebraic expressions

Discovery Activity 5 Individual

Aim: To investigate two different methods to differentiate functions involving the product of two algebraic expressions

Steps:

1. Consider the function $y = (x^2+1)(x-4)^2$.
2. Expand the expression $(x^2+1)(x-4)^2$ and then find $\frac{dy}{dx}$ by differentiating each term separately.
3. If $u = x^2+1$ and $v = (x-4)^2$, find
   (a) $\frac{du}{dx}$ and $\frac{dv}{dx}$,
   (b) $u\frac{dv}{dx} + v\frac{du}{dx}$ in terms of x.
4. Compare the methods used in step 2 and step 3. Are the answers the same? Which method will you choose? Give your reasons.

From Discovery Activity 5 results, it is shown that there are more than one method of differentiating functions involving two algebraic expressions multiplied together like the function $y = (x^2+1)(x-4)^2$. However, in cases where expansion of algebraic expressions is difficult such as $(x^2+1)\sqrt{x-4}$, the **product rule** illustrated in step 3 is often used to differentiate such functions.

In general, the formula to find the first derivative of functions involving the product of two algebraic expressions which is also known as the product rule is as follows:

If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

QR Access

To prove the product rule using the idea of limits

bit.ly/305eyTz

Excellent Tip

$\frac{d}{dx}(uv) \neq \frac{du}{dx} \times \frac{dv}{dx}$

2.2.3 2.2.4

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Discovery Activity 6 Individual

Aim: To explore two different methods to differentiate functions involving the quotient of two algebraic expressions

Steps:

1. Consider the function $y = \frac{x}{(x-1)^2}$.
2. Rewrite the function $y = \frac{x}{(x-1)^2}$ in the form $y = x(x-1)^{-2}$ and determine $\frac{dy}{dx}$ by using the product rule.
3. If $u = x$ and $v = (x-1)^2$, find
   (a) $\frac{du}{dx}$ and $\frac{dv}{dx}$,
   (b) $\frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ in terms of x.
4. Compare the methods used in steps 2 and 3. Are the answers the same?
5. Then, state the method you would like to use. Give your reasons.

From Discovery Activity 6, it is shown that apart from using the product rule in differentiating a function involving the quotient of two algebraic expressions such as $y = \frac{x}{(x-1)^2}$, a **quotient rule** illustrated in step 3 can also be used.

DISCUSSION

By using the idea of limits, prove the quotient rule.

In general, the quotient rule is stated as follows:

If u and v are functions of x, and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Excellent Tip

$\frac{d}{dx}\left(\frac{u}{v}\right) \neq \frac{\frac{du}{dx}}{\frac{dv}{dx}}$

Example 8

Differentiate each of the following expressions with respect to x.

(a) $(x^2 + 1)(x - 3)^4$ (b) $(3x + 2)\sqrt{4x - 1}$

Solution

(a) Given $y = (x^2 + 1)(x - 3)^4$.

Let $u = x^2 + 1$

and $v = (x - 3)^4$

We get $\frac{du}{dx} = 2x$

and $\frac{dv}{dx} = 4(x-3)^{4-1}\frac{d}{dx}(x-3)$

$= 4(x-3)^3$

Information Corner

The product rule and the quotient rule can be respectively written as follows:

- $\frac{d}{dx}(uv) = uv' + vu'$
- $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$

where both u and v are functions of x.

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Hence, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (x^2 + 1) \times 4(x - 3)^3 + (x - 3)^4 \times 2x$

$= 4(x^2 + 1)(x - 3)^3 + 2x(x - 3)^4$

$= 2(x - 3)^3[2(x^2 + 1) + x(x - 3)]$

$\frac{dy}{dx} = 2(x - 3)^3(3x^2 - 3x + 2)$

(b) Given $y = (3x + 2)\sqrt{4x - 1}$.

Let $u = 3x + 2$

and $v = \sqrt{4x - 1} = (4x - 1)^{\frac{1}{2}}$

We get $\frac{du}{dx} = 3$

and $\frac{dv}{dx} = \frac{1}{2}(4x - 1)^{\frac{1}{2}-1}\frac{d}{dx}(4x - 1)$

$= \frac{1}{2}(4x - 1)^{-\frac{1}{2}}(4)$

$= \frac{2}{\sqrt{4x - 1}}$

Hence, $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$= (3x + 2) \times \frac{2}{\sqrt{4x - 1}} + \sqrt{4x - 1} \times 3$

$= \frac{2(3x + 2)}{\sqrt{4x - 1}} + 3\sqrt{4x - 1}$

$= \frac{2(3x + 2) + 3(4x - 1)}{\sqrt{4x - 1}}$

$\frac{dy}{dx} = \frac{18x + 1}{\sqrt{4x - 1}}$

DISCUSSION

1. Differentiate $x(1 - x^2)^2$ with respect to x by using two different methods. Are the answers the same?
2. Given $y = 3(2x - 1)^4$, find $\frac{dy}{dx}$ by using
   (a) the chain rule,
   (b) the product rule.
   Which method would you choose?

QR Access

Check answers in Example 8 using a product rule calculator.

ggbm.at/CHfcruJC

Example 9

Given $y = x\sqrt{x + 3}$, find

(a) the expression for $\frac{dy}{dx}$

(b) the gradient of the tangent at $x = 6$

Solution

(a) Let $u = x$ and $v = \sqrt{x + 3}$.

Then, $\frac{dy}{dx} = x\frac{d}{dx}\left(\sqrt{x + 3}\right) + \sqrt{x + 3}\,\frac{d}{dx}(x)$

$= x\left(\frac{1}{2\sqrt{x + 3}}\right) + \sqrt{x + 3}$

$= \frac{x + 2(x + 3)}{2\sqrt{x + 3}}$

$\frac{dy}{dx} = \frac{3(x + 2)}{2\sqrt{x + 3}}$

(b) When $x = 6$,

$\frac{dy}{dx} = \frac{3(6 + 2)}{2\sqrt{6 + 3}}$

$= \frac{24}{6}$

$= 4$

Hence, the gradient of the tangent at $x = 6$ is 4.

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Example 10

(a) Given $y = \dfrac{2x+1}{x^2-3}$, find $\dfrac{dy}{dx}$.

(b) Given $y = \dfrac{x}{\sqrt{4x-1}}$, show that $\dfrac{dy}{dx} = \dfrac{2x-1}{\sqrt{(4x-1)^3}}$.

Solution

(a) Let $u = 2x + 1$ and $v = x^2 - 3$.

Then, $\dfrac{du}{dx} = 2$ and $\dfrac{dv}{dx} = 2x$

Therefore, $\dfrac{dy}{dx} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

$= \dfrac{(x^2-3)(2) - (2x+1)(2x)}{(x^2-3)^2}$

$= \dfrac{2x^2 - 6 - (4x^2 + 2x)}{(x^2-3)^2}$

$= \dfrac{-2x^2 - 2x - 6}{(x^2-3)^2}$

$\dfrac{dy}{dx} = \dfrac{-2(x^2 + x + 3)}{(x^2-3)^2}$

(b) $\dfrac{dy}{dx} = \dfrac{\sqrt{4x-1}\dfrac{d}{dx}(x) - x\dfrac{d}{dx}\left(\sqrt{4x-1}\right)}{\left(\sqrt{4x-1}\right)^2}$

$= \dfrac{\sqrt{4x-1} - \dfrac{2x}{\sqrt{4x-1}}}{4x-1}$

$= \dfrac{\left(\sqrt{4x-1}\right)\left(\sqrt{4x-1}\right) - 2x}{(4x-1)\sqrt{4x-1}}$

$= \dfrac{4x - 1 - 2x}{\left(4x-1\right)\left(\sqrt{4x-1}\right)}$

$= \dfrac{2x-1}{\left(4x-1\right)\left(\sqrt{4x-1}\right)}$

$\dfrac{dy}{dx} = \dfrac{2x-1}{\sqrt{(4x-1)^3}}$

Self-Exercise 2.5

1. Find $\dfrac{dy}{dx}$ for each of the following functions.

(a) $y = 4x^2(5x + 3)$ (b) $y = -2x^3(x + 1)$ (c) $y = x^2(1 - 4x)^4$

(d) $y = x^2\sqrt{1 - 2x^2}$ (e) $y = (4x - 3)(2x + 7)^6$ (f) $y = (x + 5)^3(x - 4)^4$

2. Differentiate each of the following with respect to x by using product rule.

(a) $(1 - x^2)(6x + 1)$ (b) $\left(x + \dfrac{2}{x}\right)\left(x^2 - \dfrac{1}{x}\right)$ (c) $(x^3 - 5)(x^2 - 2x + 8)$

3. Given $f(x) = x\sqrt{x-1}$, find the value of $f'(5)$.

4. Find the gradient of the tangent of the curve $y = x\sqrt{x^2+9}$ at $x = 4$.

5. Differentiate each of the following expressions with respect to x.

(a) $\dfrac{3}{2x-7}$ (b) $\dfrac{3x}{4x+6}$ (c) $\dfrac{4x^2}{1-6x}$ (d) $\dfrac{x^3+1}{2x-1}$

(e) $\dfrac{\sqrt{x}}{x+1}$ (f) $\dfrac{x}{\sqrt{x-1}}$ (g) $\dfrac{3x^2}{\sqrt{2x^2+3}}$ (h) $\sqrt{\dfrac{4x+1}{3x^2-7}}$

6. Find the value of constant r such that $\dfrac{d}{dx}\left(\dfrac{2x-3}{x+5}\right) = \dfrac{r}{(x+5)^2}$

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Formative Exercise 2.2

Quiz bit.ly/2N9zuUi

1. Differentiate each of the following expressions with respect to x.

(a) $9x^2 - \frac{3}{x^2}$ (b) $\frac{6}{x^3} - \frac{1}{x} + 8$ (c) $5x + 4\sqrt{x} - 7$ (d) $\frac{10}{\sqrt{x}} + \frac{3}{\sqrt[3]{x}}$

(e) $\left(x^2 - \frac{3}{x}\right)^2$ (f) $\frac{8x^2 + x}{\sqrt{x}}$ (g) $\frac{4}{9x^3} - \pi x + 6$ (h) $\sqrt{x}(2 - x)$

2. If $f(x) = 3x^{\frac{2}{3}} + 6x^{-\frac{1}{3}}$, find the value of $f'(8)$.

3. Given $f(t) = \frac{6t^3}{\sqrt[3]{t}}$,

(a) simplify $f(t)$, (b) find $f'(t)$, (c) find the value of $f'\left(\frac{1}{8}\right)$.

4. Given $s = 3t^2 + 5t - 7$, find $\frac{ds}{dt}$ and the range of the values of t such that $\frac{ds}{dt}$ is negative.

5. Given $\frac{dy}{dx}$ for the function $y = ax^3 + bx^2 + 3$ at the point $(1, 4)$ is 7, find the values of a and b.

6. Find the coordinates of a point for the function $y = x^3 - 3x^2 + 6x + 2$ such that $\frac{dy}{dx}$ is 3.

7. Given the function $h(x) = kx^3 - 4x^2 - 5x$, find

(a) $h'(x)$, in terms of k, (b) the value of k if $h'(1) = 8$.

8. Find $\frac{dy}{dx}$ for each of the following functions.

(a) $y = \frac{3}{4}\left(\frac{x}{6} - 1\right)^4$ (b) $y = \frac{1}{12}(10x - 3)^6$ (c) $y = \frac{8}{2 - 5x}$

(d) $y = \left(x - \frac{1}{x}\right)^3$ (e) $y = \frac{1}{\sqrt[3]{3 - 9x}}$ (f) $y = \sqrt{x^2 + 6x + 6}$

9. If $y = \frac{24}{(3x - 5)^2}$, find the value of $\frac{dy}{dx}$ when $x = 2$.

10. Find the value of constant a and constant b such that $\frac{d}{dx}\left(\frac{1}{(3x - 2)^3}\right) = -\frac{a}{(3x - 2)^b}$

11. Differentiate each of the following with respect to x.

(a) $4x(2x - 1)^5$ (b) $x^4(3x + 1)^7$ (c) $x\sqrt{x + 3}$ (d) $(x + 7)^5(x - 5)^3$

(e) $\frac{1 - \sqrt{x}}{1 + \sqrt{x}}$ (f) $\frac{x}{\sqrt{4x + 1}}$ (g) $\frac{1}{x^2 + 2x + 7}$ (h) $\frac{1 - 2x^3}{x - 1}$

12. Show that if $f(x) = x\sqrt{x^2 + 3}$, then $f'(x) = \frac{2x^2 + 3}{\sqrt{x^2 + 3}}$

13. Given $y = \frac{4x - 3}{x^2 + 1}$, find $\frac{dy}{dx}$ and determine the range of the values of x such that all the values of y and $\frac{dy}{dx}$ are positive.

14. Given $y = \frac{x - 2}{x^2 + 5}$, find the range of the values of x such that y and $\frac{dy}{dx}$ are both negative.

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2.3 The Second Derivative

Second derivative of an algebraic function

Consider the cubic function $y = f(x) = x^3 - 2x^2 + 3x - 5$.

Cubic function of x		Quadratic function of x
$y = f(x) = x^3 - 2x^2 + 3x - 5$	First derivative	$\frac{dy}{dx} = f'(x) = 3x^2 - 4x + 3$

Notice that differentiating a function $y = f(x)$ with respect to x will result in another different function of x. The function $\frac{dy}{dx}$ or $f'(x)$ is known as the first derivative of the function $y = f(x)$ with respect to x. What will happen if we want to differentiate $\frac{dy}{dx}$ or $f'(x)$ with respect to x?

When the function $\frac{dy}{dx}$ or $f'(x)$ is differentiated with respect to x, we get $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ or $\frac{d}{dx}[f'(x)]$. This function is written as $\frac{d^2y}{dx^2}$ or $f''(x)$ and is called the **second derivative** of the function $y = f(x)$ with respect to x. In general,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } f''(x) = \frac{d}{dx}[f'(x)]$$

Example 11

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the function $y = x^3 + \frac{4}{x^2}$.

(b) If $g(x) = 2x^3 + 3x^2 - 7x - 9$, find $g''\left(\frac{1}{4}\right)$ and $g''(-1)$.

Solution

(a) $y = x^3 + \frac{4}{x^2}$

$= x^3 + 4x^{-2}$

$\frac{dy}{dx} = 3x^2 - 8x^{-3}$

$\frac{dy}{dx} = 3x^2 - \frac{8}{x^3}$

$\frac{d^2y}{dx^2} = 6x + 24x^{-4}$

$\frac{d^2y}{dx^2} = 6x + \frac{24}{x^4}$

(b) $g(x) = 2x^3 + 3x^2 - 7x - 9$

$g'(x) = 6x^2 + 6x - 7$

$g''(x) = 12x + 6$

Thus, $g''\left(\frac{1}{4}\right) = 12\left(\frac{1}{4}\right) + 6$

$= 3 + 6$

$= 9$

$g''(-1) = 12(-1) + 6$

$= -12 + 6$

$= -6$

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Example 12

Given the function $f(x) = x^3 + 2x^2 + 3x + 4$, find the values of x such that $f'(x) = f''(x)$.

Solution

Given $f(x) = x^3 + 2x^2 + 3x + 4$.
Then, $f'(x) = 3x^2 + 4x + 3$ and $f''(x) = 6x + 4$.

$$f'(x) = f''(x)$$
$$3x^2 + 4x + 3 = 6x + 4$$
$$3x^2 - 2x - 1 = 0$$
$$(3x + 1)(x - 1) = 0$$
$$x = -\frac{1}{3} \text{ or } x = 1$$

Therefore, the values of x are $-\frac{1}{3}$ and 1.

Flash Quiz

If $y = 5x - 3$, find

(a) $\left(\frac{dy}{dx}\right)^2$

(b) $\frac{d^2y}{dx^2}$

Is $\left(\frac{dy}{dx}\right)^2 = \frac{d^2y}{dx^2}$? Explain.

Self-Exercise 2.6

1. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following functions.

(a) $y = 3x^4 - 5x^2 + 2x - 1$ (b) $y = 4x^2 - \frac{2}{x}$ (c) $y = (3x + 2)^8$

2. Find $f'(x)$ and $f''(x)$ for each of the following functions.

(a) $f(x) = \sqrt{x} + \frac{1}{x^2}$ (b) $f(x) = \frac{x^4 + 2}{x^2}$ (c) $f(x) = \frac{2x + 5}{x - 1}$

3. Given $y = x^3 + 3x^2 - 9x + 2$, find the possible coordinates of A where $\frac{dy}{dx} = 0$. Then, find the value of $\frac{d^2y}{dx^2}$ at point A.

Formative Exercise 2.3

Quiz bit.ly/36E4pzS

1. If $xy - 2x^2 = 3$, show that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y$.

2. Find the value of $f'(1)$ and $f''(1)$ for each of the following functions.

(a) $f(x) = 3x - 2x^3$ (b) $f(x) = x^2(5x - 3)$ (c) $f(x) = \frac{x^3 + x}{x^2}$

3. If $f(x) = \sqrt{x^2 - 5}$, find $f'(3)$ and $f''(-3)$.

4. If $a = t^3 + 2t^2 + 3t + 4$, find the values of t such that $\frac{da}{dt} = \frac{d^2a}{dt^2}$.

5. Given the function $g(x) = hx^3 - 4x^2 + 5x$. Find the value of h if $g''(1) = 4$.

6. Given $f(x) = x^3 - x^2 - 8x + 9$, find

(a) the values of x such that $f'(x) = 0$, (b) $f''(x)$,

(c) the value of x such that $f''(x) = 0$, (d) the range of x for $f''(x) < 0$.

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2.4 Application of Differentiation

The building of a roller coaster not only takes safety into consideration, but also users' maximum enjoyment out of the ride. Each point on the track is specially designed to achieve these objectives.

Which techniques do we need in order to determine the gradient at each of the points along the track of this roller coaster?

Gradient of tangent to a curve at different points

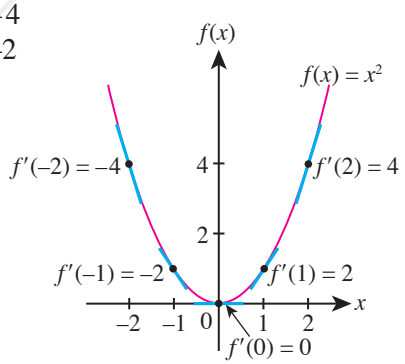
We have already learnt that the gradient of a curve at a point is also the gradient of the tangent at that point. The gradient changes at different points on a curve.

Consider the function $y = f(x) = x^2$ and its gradient function, $\frac{dy}{dx} = f'(x) = 2x$. The gradient function $f'(x)$ is used to determine the gradient of tangent to the curve at any point on the function graph $f(x)$.

For example, for the function $f(x) = x^2$:

- When $x = -2$, the gradient of the tangent, $f'(-2) = 2(-2) = -4$
- When $x = -1$, the gradient of the tangent, $f'(-1) = 2(-1) = -2$
- When $x = 0$, the gradient of the tangent, $f'(0) = 2(0) = 0$
- When $x = 1$, the gradient of the tangent, $f'(1) = 2(1) = 2$
- When $x = 2$, the gradient of the tangent, $f'(2) = 2(2) = 4$

The diagram on the right shows the gradient of tangents to the curve $f(x) = x^2$ at five different points.

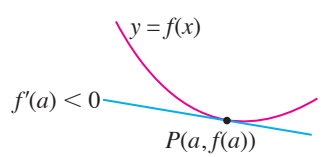


In general, the types of gradient of tangents, $f'(a)$ and the properties of a gradient of a tangent to a curve $y = f(x)$ at point $P(a, f(a))$ can be summarised as follow.

The gradient of a tangent at point $x = a$, $f'(a)$

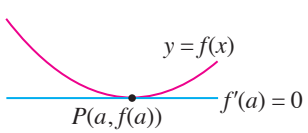
Negative gradient when $f'(a) < 0$

The tangent line slants to the left.



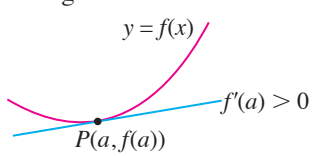
Zero gradient when $f'(a) = 0$

The tangent line is horizontal.



Positive gradient when $f'(a) > 0$

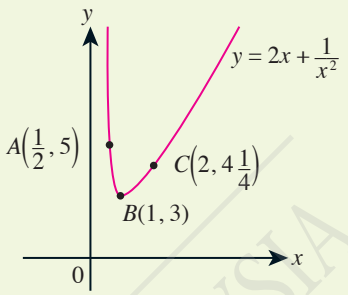
The tangent line slants to the right.



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Example 13

The diagram on the right shows a part of the curve $y = 2x + \frac{1}{x^2}$ and the points $A\left(\frac{1}{2}, 5\right)$, $B(1, 3)$ and $C\left(2, 4\frac{1}{4}\right)$ that are on the curve.



(a) Find

(i) an expression for $\frac{dy}{dx}$,

(ii) the gradient of the tangent to the curve at points A, B and C.

(b) For each of the points A, B and C, state the condition of the gradient of the tangent to the curve.

Solution

(a) (i) $y = 2x + \frac{1}{x^2}$

$= 2x + x^{-2}$

$\frac{dy}{dx} = 2 + (-2x^{-2-1})$

$= 2 - 2x^{-3}$

$\frac{dy}{dx} = 2 - \frac{2}{x^3}$

(ii) Gradient of the tangent at $A\left(\frac{1}{2}, 5\right) = 2 - \frac{2}{\left(\frac{1}{2}\right)^3}$

$= -14$

Gradient of the tangent at $B(1, 3) = 2 - \frac{2}{1^3}$

$= 0$

Gradient of the tangent at $C\left(2, 4\frac{1}{4}\right) = 2 - \frac{2}{2^3}$

$= 1\frac{3}{4}$

(b) At point A, the gradient of the tangent is -14 (< 0). Hence, the gradient is negative and the tangent line slants to the left.

At point B, the gradient of the tangent is 0. Hence, the gradient is zero and the tangent line is horizontal.

At point C, the gradient of the tangent is $1\frac{3}{4}$ (> 0). Hence, the gradient is positive and the tangent line slants to the right.

Self-Exercise 2.7

1. The equation of a curve is $y = 9x + \frac{1}{x}$ for $x > 0$.

(a) (i) Find the gradient of the tangent to the curve at $x = \frac{1}{4}$ and $x = 1$.

(ii) For each of the x-coordinates, state the condition of the gradient of the tangent to the curve.

(b) Subsequently, find the coordinates of the point where the tangent line is horizontal.

2. The curve $y = ax^2 + \frac{b}{x}$ has gradients -14 and 7 at $x = \frac{1}{2}$ and $x = 2$ respectively.

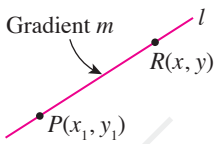
(a) Determine the values of a and b.

(b) Find the coordinates of the point on the curve where the gradient of the tangent is zero.

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The equation of tangent and normal to a curve at a point

Consider the points $P(x_1, y_1)$ and $R(x, y)$ that are on the straight line l with gradient m as shown in the diagram on the right. It is known that the gradient of $PR = \frac{y - y_1}{x - x_1} = m$.

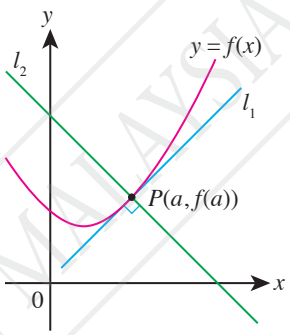


Hence, the formula for the equation of straight line l with gradient m that passes through point $P(x_1, y_1)$ can be written as:

$$y - y_1 = m(x - x_1)$$

This formula can be used to find the equation of tangent and the normal to a curve at a particular point.

In the diagram on the right, line l_1 is a **tangent** to the curve $y = f(x)$ at point $P(a, f(a))$. The gradient of the tangent for l_1 is the value of $\frac{dy}{dx}$ at $x = a$, that is, $f'(a)$.



Then, the equation of the tangent is:

$$y - f(a) = f'(a)(x - a)$$

Line l_2, which is perpendicular to tangent l_1 is the **normal** to the curve $y = f(x)$ at $P(a, f(a))$. If the gradient of the tangent, $f'(a)$ exists and is non-zero, the gradient of the normal based on the relation of $m_1 m_2 = -1$ is $-\frac{1}{f'(a)}$.

Then, the equation of the normal is:

$$y - f(a) = -\frac{1}{f'(a)}(x - a)$$

Example 14

Find the equation of the tangent and normal to the curve $f(x) = x^3 - 2x^2 + 5$ at point $P(2, 5)$.

Solution

Given $f(x) = x^3 - 2x^2 + 5$, so $f'(x) = 3x^2 - 4x$.

When $x = 2$, $f'(2) = 3(2)^2 - 4(2) = 12 - 8 = 4$

Gradient of the tangent at point $P(2, 5)$ is 4.

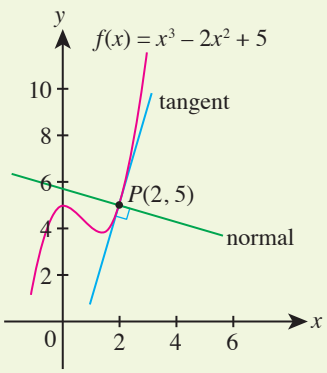
Equation of the tangent is $y - 5 = 4(x - 2)$

$$y - 5 = 4x - 8$$
$$y = 4x - 3$$

Gradient of the normal at point $P(2, 5)$ is $-\frac{1}{4}$.

Equation of the normal is $y - 5 = -\frac{1}{4}(x - 2)$

$$4y - 20 = -x + 2$$
$$4y + x = 22$$



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Self-Exercise 2.8

1. Find the equation of the tangent and normal to the following curves at the given points.

(a) $f(x) = 5x^2 - 7x - 1$ at the point $(1, -3)$ (b) $f(x) = x^3 - 5x + 6$ at the point $(2, 4)$

(c) $f(x) = \sqrt{2x + 1}$ at the point $(4, 3)$ (d) $f(x) = \dfrac{x + 1}{x - 1}$ at the point $(3, 2)$

2. Find the equation of the tangent and normal to the following curves at the given value of x.

(a) $y = 2x^3 - 4x + 3, x = 1$ (b) $y = \sqrt{x} - \dfrac{1}{\sqrt{x}}, x = 4$ (c) $y = \sqrt{x + 1}, x = 3$

(d) $y = \dfrac{5}{x^2 + 1}, x = -2$ (e) $y = 2 + \dfrac{1}{x}, x = -1$ (f) $y = \dfrac{x^2 + 3}{x + 1}, x = 3$

3. A tangent and a normal is drawn to the curve $y = x\sqrt{1 - 2x}$ at $x = -4$. Find

(a) the value of $\dfrac{dy}{dx}$ at $x = -4$, (b) the equation of the tangent,

(c) the equation of the normal.

4. (a) The tangent to the curve $y = (x - 2)^2$ at the point $(3, 1)$ passes through $(k, 7)$. Find the value of k.

(b) The normal to the curve $y = 7x - \dfrac{6}{x}$ at $x = 1$ intersects the x-axis at A. Find the coordinates of A.

Solving problems involving tangent and normal

Diagram 2.1(a) shows a circular pan where a quarter of it has been cut off, that is, AOB has been removed. A ball circulates along the circumference of the pan.

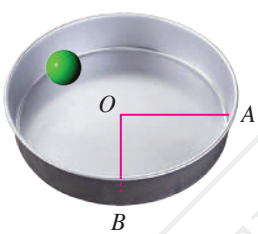


Diagram 2.1(a)

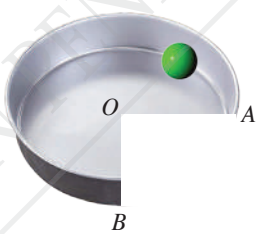


Diagram 2.1(b)

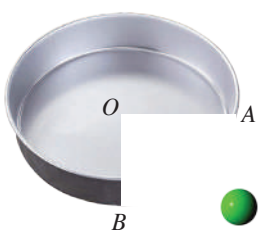


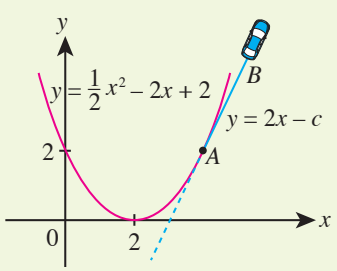
Diagram 2.1(c)

What will happen to the movement of the ball when it reaches point A where that quarter portion AOB has been removed as shown in Diagram 2.1(b)? Will the ball move tangential to the circumference of the pan at A?

Example 15 MATHEMATICAL APPLICATIONS

The diagram on the right shows a road which is represented by the curve $y = \dfrac{1}{2}x^2 - 2x + 2$. Kumar drove on the road. As it was raining and the road was slippery, his car skidded at A and followed the line AB, which is tangent to the road at A and has an equation of $y = 2x - c$. Find

(a) the coordinates of A, (b) the value of constant c.



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Solution

1 . Understanding the problem

- The road is represented by the curve $y = \frac{1}{2}x^2 - 2x + 2$.
- Kumar drove on the road and skidded at point A and then followed the path $y = 2x - c$, which is the tangent to the road.
- Find the coordinates of A and the value of constant c.

2 . Planning the strategy

- Find the gradient function, $\frac{dy}{dx}$ of the curve $y = \frac{1}{2}x^2 - 2x + 2$.
- The gradient for $y = 2x - c$ is 2.
- Solve $\frac{dy}{dx} = 2$ to get the coordinates of A.
- Substitute the coordinates of A obtained into the function $y = 2x - c$ to obtain the value of constant c.

3 . Implementing the strategy

(a) $y = \frac{1}{2}x^2 - 2x + 2$

$\frac{dy}{dx} = x - 2$

Since $y = 2x - c$ is the tangent to the road $y = \frac{1}{2}x^2 - 2x + 2$ at point A, so

$\frac{dy}{dx} = 2$

$x - 2 = 2$

$x = 4$

Since point A lies on the curve, so

$y = \frac{1}{2}(4)^2 - 2(4) + 2$

$y = 2$

Then, the coordinates of A is $(4, 2)$.

(b) The point $A(4, 2)$ lies on the line AB, that is $y = 2x - c$, then

$2 = 2(4) - c$

$c = 6$

Hence, the value of constant c is 6.

4 . Check and reflect

(a) Substitute $x = 4$ from $A(4, 2)$ into $y = 2x - 6$, and we obtain

$y = 2(4) - 6$

$y = 8 - 6$

$y = 2$

(b) The path AB, that is, $y = 2x - c$ whose gradient is 2 passes through the point $A(4, 2)$ and $(0, -c)$, then the gradient of $AB = 2$

$\frac{y_2 - y_1}{x_2 - x_1} = 2$

$\frac{2 - (-c)}{4 - 0} = 2$

$\frac{2 + c}{4} = 2$

$c + 2 = 8$

$c = 8 - 2$

$c = 6$

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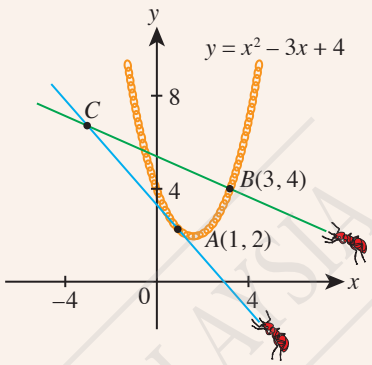
Self-Exercise 2.9

1. The diagram on the right shows a bracelet which is represented by the curve $y = x^2 - 3x + 4$ where point $A(1, 2)$ and point $B(3, 4)$ are located on the bracelet. The line AC is a tangent to the bracelet at point A and the line BC is a normal to the bracelet at point B. Two ants move along AC and BC, and meet at point C. Find

(a) the equation of the tangent at point A,

(b) the equation of the normal at point B,

(c) the coordinates of C where the two ants meet.



2. The equation of a curve is $y = 2x^2 - 5x - 2$.

(a) Find the equation of a normal to the curve at point $A(1, -5)$.

(b) The normal meets the curve again at point B. Find the coordinates of B.

(c) Subsequently, find the coordinates of the midpoint of AB.

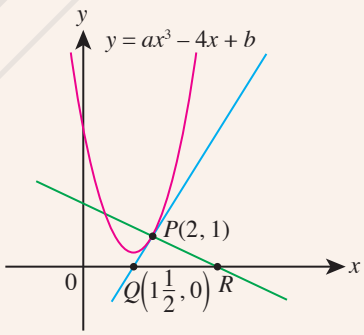
3. In the diagram on the right, the tangent to the curve $y = ax^3 - 4x + b$ at $P(2, 1)$ intersects the x-axis at $Q\left(1\frac{1}{2}, 0\right)$. The normal at P intersects the x-axis at R. Find

(a) the values of a and b,

(b) the equation of the normal at point P,

(c) the coordinates of R,

(d) the area of triangle PQR.



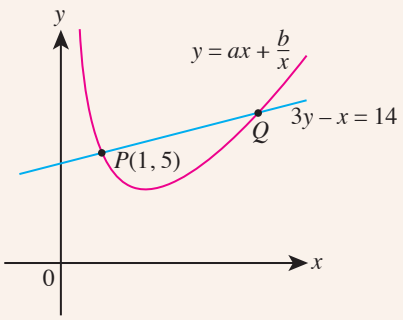
4. The diagram on the right shows a part of the curve $y = ax + \frac{b}{x}$. The line $3y - x = 14$ is a normal to the curve at $P(1, 5)$ and this normal intersects the curve again at Q. Find

(a) the values of a and b,

(b) the equation of tangent at point P,

(c) the coordinates of Q,

(d) the coordinates of the midpoint of PQ.



5. (a) The tangent to the curve $y = \sqrt{2x + 1}$ at point $A(4, 3)$ intersects the x-axis at point B. Find the distance of AB.

(b) The tangent to the curve $y = hx^3 + kx + 2$ at $\left(1, \frac{1}{2}\right)$ is parallel to the normal to the curve $y = x^2 + 6x + 4$ at $(-2, -4)$. Find the value of constants h and k.

KEMENTERIAN PENDIDIKAN MALAYSIA

Turning points and their nature

There are three types of stationary points, that is maximum point, minimum point and point of inflection. Amongst the stationary points, which are turning points and which are not turning points? Let's explore how to determine the turning points and their nature.

Discovery Activity 7

Group 21st cl STEM CT

Aim: To determine turning points on a function graph and their nature by observing the neighbouring gradients about those turning points

ggbm.at/cygujkvm

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Pay attention to the graph $y = -x^2 + 2x + 3$ and the tangent to the curve at point P shown on the plane.
3. Drag point P along the curve and observe the gradient of the curve at point P.
4. Then, copy and complete the following table.

x*-coordinates at *P	−1	0	1	2	3
Gradient of the curve at point *P*, $\frac{dy}{dx}$	4				
Sign for $\frac{dy}{dx}$	+				
Sketch of the tangent	/				
Sketch of the graph					

5. Substitute the values of a, b and c into the function $f(x) = ax^2 + bx + c$ to obtain the graph for the curve $y = x^2 + 2x - 3$. Repeat steps 3 and 4 by substituting the x-coordinates from point P in the table with $x = -3, -2, -1, 0$ and 1.
6. Click on $f(x) = ax^2 + bx + c$ one more time and change x^2 to x^3. Then, substitute the values of a, b and c to get the curve $y = x^3 + 4$. Repeat steps 3 and 4 by substituting x-coordinates for point P in the table with $x = -2, -1, 0, 1$ and 2.
7. For each of the following functions that was investigated:
   (a) $y = -x^2 + 2x + 3$ (b) $y = x^2 + 2x - 3$ (c) $y = x^3 + 4$
   (i) State the coordinates of the stationary points.
   (ii) When x increases through the stationary points, how do the values of $\frac{dy}{dx}$ change?
   (iii) What can you observe on the signs of the gradients for each curve?
   (iv) Determine the types and nature of the stationary points.
8. Present your findings to the class and have a Q and A session among yourselves.

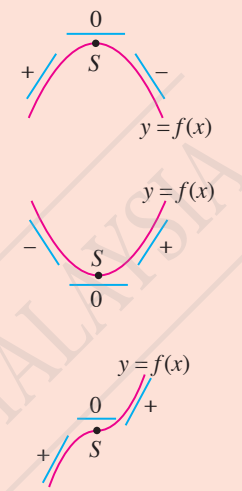
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From Discovery Activity 7, a stationary point can be determined when $\frac{dy}{dx} = 0$ and their nature can be summarised as follows:

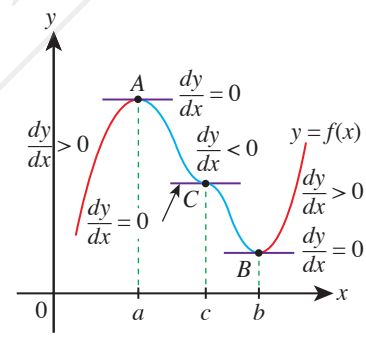
For a curve $y = f(x)$ with a stationary point S at $x = a$,

- If the sign of $\frac{dy}{dx}$ changes from positive to negative as x increases through a, then point S is a maximum point.
- If the sign of $\frac{dy}{dx}$ changes from negative to positive when x increases through a, then point S is a minimum point.
- If the sign of $\frac{dy}{dx}$ does not change as x increases through a, then point S is a **point of inflection**.

A stationary point is known as a **turning point** if the point is a maximum or minimum point.



Consider the graph of a function $y = f(x)$ as shown in the diagram on the right. Based on the diagram, the increasing function graph which is red has a positive gradient, that is $\frac{dy}{dx} > 0$ while the decreasing function graph which is blue has a negative gradient, that is $\frac{dy}{dx} < 0$.



The points with $f'(x) = \frac{dy}{dx} = 0$ are called the **stationary points** where tangents to the graph at those points are horizontal. Hence, those points A, B and C are stationary points for $y = f(x)$.

From the graph $y = f(x)$ on the right, it is found that:

The stationary point at A is the maximum point

When x increases through $x = a$, the value of $\frac{dy}{dx}$ changes sign from positive to negative.

The stationary point at B is the minimum point

When x increases through $x = b$, the value of $\frac{dy}{dx}$ changes sign from negative to positive.

The maximum point A and the minimum point B are called **turning points**. At the stationary point C, the value of $\frac{dy}{dx}$ does not change in sign as x increases through $x = c$. The stationary point C is not a turning point. This stationary point which is not a maximum or a minimum point is called **point of inflection**, that is, a point on the curve at which the curvature of the graph changes.

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Example 16

Given the curve $y = x^3 - 3x^2 - 9x + 11$,
(a) find the coordinates of the turning points of the curve.
(b) determine whether each of the turning points is a maximum or minimum point.

Solution

(a) $y = x^3 - 3x^2 - 9x + 11$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$
$$= 3(x^2 - 2x - 3)$$
$$\frac{dy}{dx} = 3(x + 1)(x - 3)$$

For a turning point, $\frac{dy}{dx} = 0$

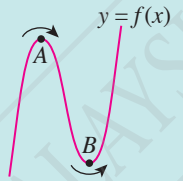
$$3(x + 1)(x - 3) = 0$$
$$x = -1 \text{ or } x = 3$$

When $x = -1$, $y = (-1)^3 - 3(-1)^2 - 9(-1) + 11$
$y = 16$

When $x = 3$, $y = 3^3 - 3(3)^2 - 9(3) + 11$
$y = -16$

Thus, the turning points are $(-1, 16)$ and $(3, -16)$.

Information Corner



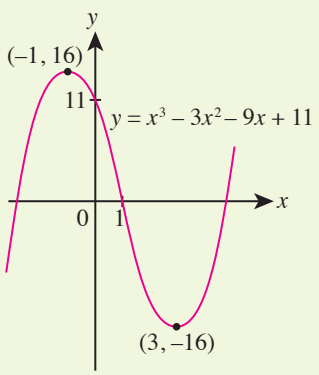
When the curve $y = f(x)$ turns and changes direction at points A and B, the maximum point A dan the minimum point B are called turning points.

(b)

x	−1.5	−1	−0.5	2.5	3	3.5
$\frac{dy}{dx}$	6.75	0	−5.25	−5.25	0	6.75
Sign for $\frac{dy}{dx}$	+	0	−	−	0	+
Sketch of the tangent	↗	—	↘	↘	—	↗
Sketch of the graph	∩			∪		

From the table, the sign for $\frac{dy}{dx}$ changes from positive to negative when x increases through $x = -1$ and the sign for $\frac{dy}{dx}$ changes from negative to positive as x increases through $x = 3$. Hence, the turning point $(-1, 16)$ is a maximum point while the turning point $(3, -16)$ is a minimum point.

The graph on the right is a sketch of the curve $y = x^3 - 3x^2 - 9x + 11$ with the turning point $(-1, 16)$ as its maximum point and the turning point $(3, -16)$ as its minimum point.



Besides the **sketching of tangents** method for a function $y = f(x)$, **second order derivative**, $\frac{d^2y}{dx^2}$ whenever possible can also be used to determine whether a turning point is a maximum or minimum point.

Diagram 2.2 shows the graph for the curve $y = 3x - x^3$ with the turning point at $P(1, 2)$ and also its gradient function graph, $\frac{dy}{dx} = 3 - 3x^2$.

From the graph $\frac{dy}{dx}$ against x, notice that:

$\frac{dy}{dx}$ decreases as x increases through $x = 1$
$\Leftrightarrow$ The rate of change of $\frac{dy}{dx}$ is negative at $x = 1$
$\Leftrightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ at $x = 1$

Hence, the turning point $P(1, 2)$ with $\frac{dy}{dx} = 0$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right) < 0$ is a maximum point.

In general,

A turning point on a curve $y = f(x)$ is a maximum point when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

Diagram 2.3 shows the graph for the curve $y = x + \frac{4}{x} - 2$ with the turning point at $P(2, 2)$ and its gradient function graph, $\frac{dy}{dx} = 1 - \frac{4}{x^2}$.

From the graph $\frac{dy}{dx}$ against x, notice that:

$\frac{dy}{dx}$ increases when x increases through $x = 2$
$\Leftrightarrow$ The rate of change of $\frac{dy}{dx}$ is positive at $x = 2$
$\Leftrightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$ at $x = 2$

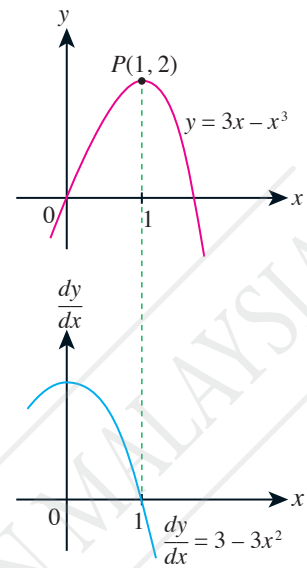


Diagram 2.2

Information Corner

- Sketching of tangents method is used to determine the nature of stationary points.
- Second order derivative is used to determine the nature of turning points.

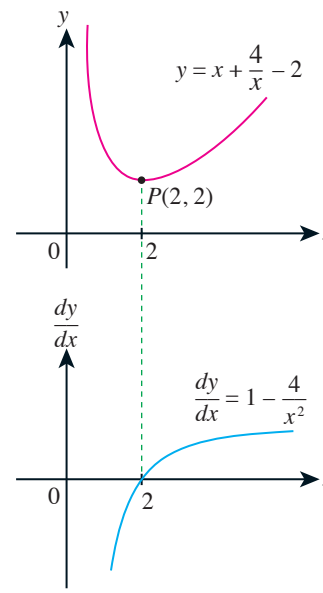


Diagram 2.3

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Hence, the turning point $P(2, 2)$ with $\frac{dy}{dx} = 0$ and $\frac{d}{dx}\left(\frac{dy}{dx}\right) > 0$ is a minimum point.

In general,

A turning point on a curve $y = f(x)$ is a minimum point when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$.

Example 17

Find the stationary points for each of the following curves and determine the nature of each stationary point.

(a) $y = 2x^3 + 3x^2 - 12x + 5$ (b) $y = x^4 - 4x^3 + 1$

Solution

(a) $y = 2x^3 + 3x^2 - 12x + 5$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

$= 6(x^2 + x - 2)$

$\frac{dy}{dx} = 6(x + 2)(x - 1)$

For stationary points, $\frac{dy}{dx} = 0$

$6(x + 2)(x - 1) = 0$

$x = -2$ or $x = 1$

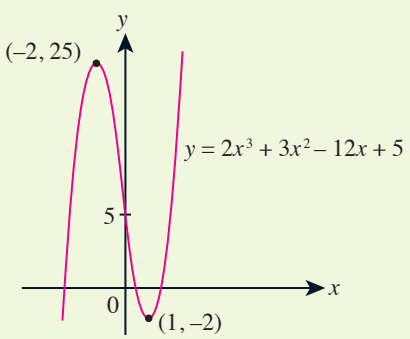
When $x = -2$, $y = 2(-2)^3 + 3(-2)^2 - 12(-2) + 5$

$y = 25$

When $x = 1$, $y = 2(1)^3 + 3(1)^2 - 12(1) + 5$

$y = -2$

Thus, the stationary points are $(-2, 25)$ and $(1, -2)$.



$\frac{d^2y}{dx^2} = 12x + 6$

When $x = -2$, $\frac{d^2y}{dx^2} = 12(-2) + 6 = -18 < 0$

When $x = 1$, $\frac{d^2y}{dx^2} = 12(1) + 6 = 18 > 0$

Hence, $(-2, 25)$ is a maximum point and $(1, -2)$ is a minimum point.

(b) $y = x^4 - 4x^3 + 1$

$\frac{dy}{dx} = 4x^3 - 12x^2$

$\frac{dy}{dx} = 4x^2(x - 3)$

For stationary point, $\frac{dy}{dx} = 0$

$4x^2(x - 3) = 0$

$x = 0$ or $x = 3$

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When $x = 0$, $y = 0^4 - 4(0)^3 + 1 = 1$
When $x = 3$, $y = 3^4 - 4(3)^3 + 1 = -26$
Thus, the stationary points are $(0, 1)$ and $(3, -26)$.

$\frac{d^2y}{dx^2} = 12x^2 - 24x$

When $x = 0$, $\frac{d^2y}{dx^2} = 12(0)^2 - 24(0) = 0$

x	-0.1	0	0.1
$\frac{dy}{dx}$	-0.124	0	-0.116
Sign for $\frac{dy}{dx}$	$-$	0	$-$
Sketch of the tangent	↘	—	↘
Sketch of the graph			

From the table, we see $\frac{dy}{dx}$ changing from negative to zero and then to negative again, that is, no change in signs as x increases through 0. Therefore, $(0, 1)$ is a point of inflection.

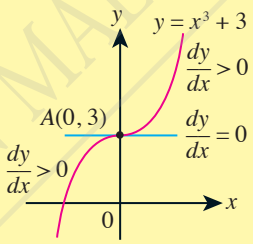
When $x = 3$, $\frac{d^2y}{dx^2} = 12(3)^2 - 24(3) = 36 > 0$

Then, $(3, -26)$ is a minimum point.

Excellent Tip

When $\frac{d^2y}{dx^2} = 0$, the tangent sketching method is used to determine the nature of the stationary point.

DISCUSSION



In the above diagram, point A is neither a maximum nor a minimum point for the function $y = x^3 + 3$, but is called a point of inflection.
Can you give three other examples of function that have a point of inflection?

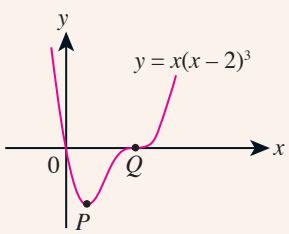
Self-Exercise 2.10

1. Find the coordinates of the turning points for each of the following curves. In each case, determine whether the turning points are maximum or minimum points.

(a) $y = x^3 - 12x$ (b) $y = x(x - 6)^2$ (c) $y = x\sqrt{18 - x^2}$ (d) $y = (x - 6)(4 - 2x)$

(e) $y = x + \frac{4}{x}$ (f) $y = x^2 + \frac{1}{x^2}$ (g) $y = x + \frac{1}{x - 1}$ (h) $y = \frac{(x - 3)^2}{x}$

2. The diagram on the right shows a part of the curve $y = x(x - 2)^3$.



(a) Find an expression for $\frac{dy}{dx}$.

(b) Find the coordinates of the two stationary points, P and Q.

(c) Subsequently, determine the nature of stationary point Q by using the tangent sketching method.

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Solving problems involving maximum and minimum values and interpreting the solutions

A lot of containers for food and beverages in the market are cylindrical in shape. How do the food and beverage tin manufacturers determine the size of the tin so that the cost of production is at a minimum?

Can the first and second order derivatives assist the manufacturers in solving this problem?

Example 18 MATHEMATICAL APPLICATIONS

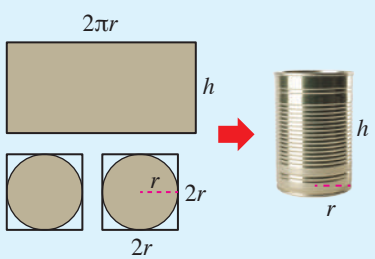


A factory wants to produce cylindrical tins from aluminium sheets to contain food. Each tin has a volume of 512 cm^3. The curved surface is made by rolling a rectangular piece of aluminium while the top and bottom are circular pieces cut out from two aluminium squares. Find the radius of the tin, in cm, such that the total surface of the aluminium sheets used will be minimum.

Solution

1 . Understanding the problem

- Let r cm be the radius of the base and h cm be the height of the tin.
- Volume of the tin, $V = \pi r^2 h = 512$ cm^3
- Total surface area of the aluminium sheets used,
  $A = 2(2r)^2 + 2\pi rh$
  $A = 2(4r^2) + 2\pi rh$
  $A = 8r^2 + 2\pi rh$
- Find the value of r such that A is minimum.



2 . Planning the strategy

- Express A in terms of one of the variables, that is, express h in terms of r.
- Find the value of r when $\frac{dA}{dr} = 0$.
- Using the value of r obtained, determine whether A is maximum or minimum.

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3 . Implementing the strategy

Volume of the tin, $V = 512$

$$\pi r^2 h = 512$$

$$h = \frac{512}{\pi r^2} \quad \ldots (1)$$

Total surface area, A cm^2, of the aluminium sheets used is given by

$$A = 8r^2 + 2\pi rh \quad \ldots (2)$$

Substitute (1) into (2),

$$A = 8r^2 + 2\pi r\left(\frac{512}{\pi r^2}\right)$$

$$A = 8r^2 + \frac{1\,024}{r}$$

$$\frac{dA}{dr} = 16r - \frac{1\,024}{r^2}$$

To obtain minimum value,

$$\frac{dA}{dr} = 0$$

$$16r - \frac{1\,024}{r^2} = 0$$

$$16r^3 - 1\,024 = 0$$

$$r^3 = \frac{1\,024}{16}$$

$$r^3 = 64$$

$$r = \sqrt[3]{64}$$

$$r = 4$$

$$\frac{dA}{dr} = 16r - 1\,024r^{-2}$$

$$\frac{d^2A}{dr^2} = 16 + \frac{2\,048}{r^3}$$

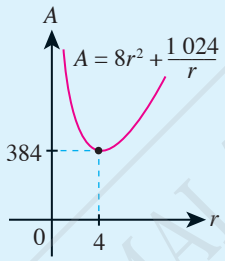
When $r = 4$, $\frac{d^2A}{dr^2} = 16 + \frac{2\,048}{4^3}$

$$= 48 > 0$$

Hence, A is minimum when the radius of the base circle is 4 cm.

4 . Check, reflect and interpret

Sketch a graph $A = 8r^2 + \frac{1\,024}{r}$ to show that the value of A has a minimum at $r = 4$.



Therefore, the factory needs to produce food tins with base radius 4 cm and with height, $h = \frac{512}{\pi r^2}$

$= \frac{512}{\pi(4)^2} = 10.186$ cm so that the total surface area of the aluminium sheets used will be minimum.

Flash Quiz

From the two equations obtained in Example 18,
$\pi r^2 h = 512$ ... (1)
$A = 8r^2 + 2\pi rh$ ... (2)
For equation (1), can we express r in terms of h and then substitute it into (2) to solve the problem in Example 18? Discuss.

Self-Exercise 2.11

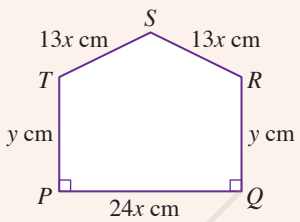
1. A wire of length 80 cm is bent to form a sector POQ of a circle with centre O. It is given that $OQ = r$ cm and $\angle POQ = \theta$ radian.

(a) Show that the area, A cm^2, of the sector POQ is $A = \frac{1}{2}r(80 - 2r)$.

(b) Then, find the maximum area of the sector POQ.

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2. A piece of wire of length 240 cm is bent to make a shape as shown in the diagram on the right.



(a) Express y in terms of x.

(b) Show that the area, A cm^2, enclosed by the wire is $A = 2\,880x - 540x^2$.

(c) Find

(i) the values of x and y for A to be maximum,

(ii) the maximum area enclosed by the wire in cm^2.

3. A factory produces cylindrical closed containers for drinks. Each container has a volume of 32π cm^3. The cost of the material used to make the top and bottom covers of the container is 2 cents per cm^2 while the cost of the material to make the curved surface is 1 cent per cm^2.

(a) Show that the cost, C to make a cylindrical drink container is $C = 4\pi r^2 + \frac{64\pi}{r}$, with r as the base radius of a cylinder.

(b) Find the dimensions of each container produced in order for the cost to be minimum.

Interpreting and determining rates of change for related quantities

Discovery Activity 8

Group | 21st cl

Aim: To investigate the rate of change of the depth of water from a depth-time graph

Steps:

1. Consider two containers, one is a cylindrical container and the other a cone container, that are to be filled with water from a pipe at a constant rate of 3π cm^{3}s^{-1}. The height of each container is 9 cm and has a volume of 48π cm^3.
2. Determine the time, t, in seconds, taken to fully fill each container.
3. Based on the surface area of the water in each container, sketch a depth-time graph to show the relation between the depth of water, h cm, with the time taken, t seconds, to fill up both containers.
4. Observe the graphs obtained. Then, answer the following questions.
   (a) Based on the gradient of each graph, determine the rate of change of depth of the water at a certain time for each container.
   (b) Did the depth of water in the cylindrical container increase at a constant rate as the container is being filled up? What about the cone? Did the rate of change of depth change as the cone is being filled up?
5. Present your group findings to the class.

From Discovery Activity 8, it is found that the rate of change of depth of water, $\frac{dh}{dt}$ at a certain time, t is the gradient of the curve at t, assuming that the water flowed into the containers at a constant rate. The rate of change can be obtained by drawing a tangent to the curve at t or by using differentiation to find the gradient of the tangent at t. The concept of **chain rule** can be applied to solve this problem easily.

2.4.5 2.4.6

KPM

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Take for example, if two variables, y and x change with time, t and are related by the equation $y = f(x)$, then the rates of change $\frac{dy}{dt}$ and $\frac{dx}{dt}$ can be related by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \text{ (Chain rule)}$$

Consider the curve $y = x^2 + 1$. If x increases at a constant rate of 2 units per second, that is, $\frac{dx}{dt} = 2$, then the rate of change of y is given by:

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt} \quad \leftarrow \text{Chain rule}$$
$$= 2x \times 2$$
$$= 4x$$

When $x = 2$, $\frac{dy}{dt} = 4(2) = 8$

Thus, the rate of change of y is 8 units per second and y is said to increase at a rate of 8 units per second when $x = 2$.

When $x = -2$, $\frac{dy}{dt} = 4(-2) = -8$

Thus, the rate of change of y is -8 units per second and y is said to decrease at a rate of 8 units per second when $x = -2$.

Example 19

A curve has an equation $y = x^2 + \frac{4}{x}$. Find

(a) an expression for $\frac{dy}{dx}$,

(b) the rate of change of y when $x = 1$ and $x = 2$, given that x increases at a constant rate of 3 units per second.

Solution

(a) $y = x^2 + \frac{4}{x}$

$= x^2 + 4x^{-1}$

$\frac{dy}{dx} = 2x - 4x^{-2}$

$\frac{dy}{dx} = 2x - \frac{4}{x^2}$

(b) When $x = 1$, $\frac{dy}{dx} = 2(1) - \frac{4}{1^2}$

$= -2$

The rate of change of y is given where

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$
$$= -2 \times 3$$
$$= -6$$

Thus, the rate of change of y is -6 units per second.

Therefore, y is said to decrease 6 units per second.

Excellent Tip

- $\frac{dy}{dx}$ is the rate of change of y with respect to x.
- $\frac{dy}{dt}$ is the rate of change of y with respect to t.
- $\frac{dx}{dt}$ is the rate of change of x with respect to t.

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When $x = 2$, $\frac{dy}{dx} = 2(2) - \frac{4}{2^2}$

$= 3$

The rate of change of y is given where

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

$= 3 \times 3$

$= 9$

Thus, the rate of change of y is 9 units per second.

Therefore, y is said to increase at a rate of 9 units per second.

Excellent Tip

If the rate of change of y over time is negative, for example $\frac{dy}{dt} = -6$, then y is said to decrease at a rate of 6 units s^{-1}, that is, its decreasing rate is 6 units s^{-1}.

Self-Exercise 2.12

1. For each of the following equations relating x and y, if the rate of change of x is 2 units per second, find the rate of change of y at the given instant.

(a) $y = 3x^2 - 4, x = \frac{1}{2}$ (b) $y = 2x^2 + \frac{1}{x}, x = 1$ (c) $y = \frac{2}{(3x-5)^3}, x = 2$

(d) $y = (4x - 3)^5, x = \frac{1}{2}$ (e) $y = \frac{x}{x+1}, y = 2$ (f) $y = x^3 + 2, y = 10$

2. For each of the following equations relating x and y, if the rate of change of y is 6 units per second, find the rate of change of x at the given instant.

(a) $y = x^3 - 2x^2, x = 1$ (b) $y = x^2 + \frac{4}{x}, x = 2$ (c) $y = \frac{2x^2}{x-1}, x = 3$

(d) $y = (x - 6)\sqrt{x-1}, x = 2$ (e) $y = \frac{2x-1}{x+1}, y = 3$ (f) $y = \sqrt{2x+7}, y = 3$

3. A curve has an equation $y = (x - 8)\sqrt{x+4}$. Find

(a) an expression for $\frac{dy}{dx}$,

(b) the rate of change of y when $x = 5$, if x increases at a rate of 6 units per second.

Solving problems involving rates of change for related quantities and interpreting the solutions

The mass, M, in kg, of a round watermelon is related to its radius, r cm, by an equation $M = \frac{2}{625}r^3$. Assume that the rate of change of radius is 0.1 cm per day when the radius is 10 cm on a particular day.



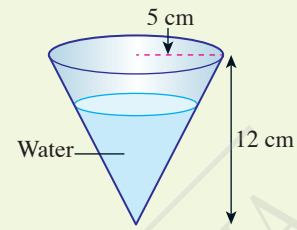
With the help of the chain rule, which relates the mass, $\frac{dM}{dt}$ to the radius, $\frac{dr}{dt}$ of the watermelon, can you find the rate of change of the mass of the watermelon on that particular day?

2.4.6 2.4.7

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Example 20

The diagram on the right shows an inverted cone with a base radius of 5 cm and a height of 12 cm filled with some water. Water leaks out from a small hole at the tip of the cone at a constant rate of 4 cm^3s^{-1}. Find the rate of change of the depth of water in the cone when the height of water is 3 cm, correct to four significant figures.

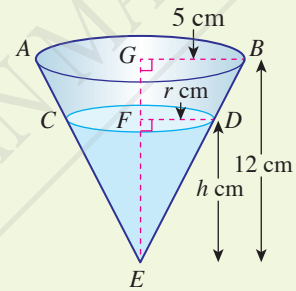


Solution

Let r cm, h cm and V cm be the radius, height and volume of the water in the cone respectively at the time t second.

Then, $V = \frac{1}{3}\pi r^2 h \ldots$ ❶

The two triangles ΔDFE and ΔBGE are similar.



Thus, $\frac{r}{5} = \frac{h}{12}$

$r = \frac{5h}{12}$ … ❷

Substitute ❷ into ❶:

$$V = \frac{1}{3}\pi\left(\frac{5h}{12}\right)^2 h$$
$$= \frac{1}{3}\pi\left(\frac{25h^2}{144}\right)h$$
$$= \frac{1}{3}\pi\left(\frac{25h^3}{144}\right)$$
$$V = \frac{25\pi}{432}h^3$$

The rate of change of V is given by the chain rule below.

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$
$$= \frac{d}{dh}\left(\frac{25\pi}{432}h^3\right) \times \frac{dh}{dt}$$
$$\frac{dV}{dt} = \frac{25\pi}{144}h^2 \times \frac{dh}{dt}$$

When $h = 3$ and $\frac{dV}{dt} = -4$, we get

$$-4 = \frac{25\pi}{144}(3)^2 \times \frac{dh}{dt}$$

V decreases, then $\frac{dV}{dt}$ is negative

$$-4 = \frac{25\pi}{16} \times \frac{dh}{dt}$$
$$\frac{dh}{dt} = -\frac{64}{25\pi}$$
$$= -0.8148$$

Hence, the rate of change of the depth of water in the cone is -0.8148 cms^{-1}. The depth of the water is said to reduce at a rate of 0.8148 cms^{-1}.

DISCUSSION

Discuss the following problem with your friends.

Water flows into a similar inverted cone shaped tank with base radius 8 cm and a height of 16 cm at a rate of 64π cm^3s^{-1}.

Let's assume h cm is the depth of the water and V cm^3 is the volume of water in the cone. Find the rate of change of

(a) the depth of water,

(b) the surface area of the water,

when the depth of water is 8 cm.

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Example 21 MATHEMATICAL APPLICATIONS

The radius of a spherical balloon filled with air increases at a rate of 0.5 cm per second. Find the rate of change of its volume when the radius is 4 cm, correct to four significant figures.



Solution

1 . Understanding the problem

- The radius of a balloon being filled with air increases at a rate of 0.5 cm per second.
- Find the rate of change of volume of the balloon when the radius is 4 cm.

2 . Planning the strategy

- Let r cm and V cm^3 be the radius and the volume of the balloon respectively at time, t second.
- Form an equation relating the volume, V to the radius, r of the balloon.
- Use the chain rule to relate the rate of change of volume to the rate of change of the radius of the balloon.

3 . Implementing the strategy

Let $V = f(r)$.

The rate of change of volume V is given:

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$

It is known that $V = \frac{4}{3}\pi r^3$.

So, $$\frac{dV}{dt} = \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right) \times \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \times \frac{dr}{dt}$$

When $r = 4$ and $\frac{dr}{dt} = 0.5$, then

$$\frac{dV}{dt} = 4\pi(4)^2 \times 0.5$$
$$= 4\pi(16) \times 0.5$$
$$= 64\pi \times 0.5$$
$$= 32\pi$$
$$= 32(3.142)$$
$$= 100.5$$

Thus, the rate of change of the volume of the balloon when the radius is $r = 4$ cm is 100.5 cm^3 per second.

4 . Check, reflect and interpret

When $\frac{dV}{dt} = 100.5$ and $\frac{dr}{dt} = 0.5$, then

$$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$
$$100.5 = 4\pi r^2 \times 0.5$$
$$100.5 = 2\pi r^2$$
$$r^2 = \frac{100.5}{2\pi}$$
$$r^2 = \frac{100.5}{2(3.142)}$$
$$r^2 = 15.993$$
$$r = \sqrt{15.993}$$
$$r = \pm 4$$

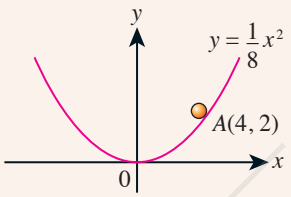
Thus, $r = 4$ cm.

So, when $r = 4$ and $\frac{dV}{dt} = 100.5$, it means that when the radius of the balloon is 4 cm, its volume increases at the rate of 100.5 cm^3 per second.

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Self-Exercise 2.13

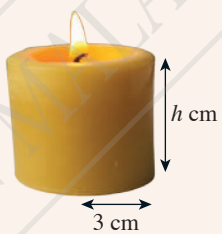
1. The diagram on the right shows a bead moving along a curve with the equation $y = \frac{1}{8}x^2$. At $A(4, 2)$, the rate of change of x is 3 units s^{-1}. Find the rate of change of the corresponding y.



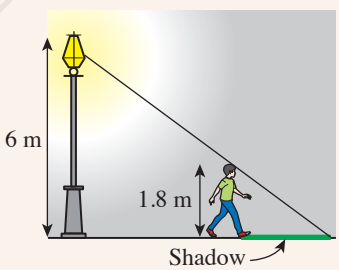
2. The area of a square with side x cm increases at a rate of 8 cm^2s^{-1}. Find the rate of change of its side when the area is 4 cm^2.

3. A block of ice in the form of a cube with sides x cm is left to melt at a rate of 10.5 cm^3 per minute. Find the rate of change of x when $x = 10$ cm.

4. The diagram on the right shows a cylindrical candle with radius 3 cm. The height is h cm and its volume is V cm^3. The candle is lit and the height decreases at a rate of 0.6 cm per minute.
(a) Express V in terms of h.
(b) Find the rate of change of the volume of the candle when its height is 8 cm.

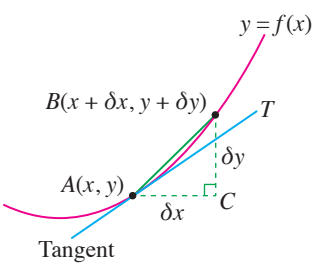


5. Chandran walks at a rate of 3.5 ms^{-1} away from a lamp post one night as shown in the diagram on the right. The heights of Chandran and the lamp post are 1.8 m and 6 m respectively. Find the rate of change of
(a) Chandran's shadow,
(b) the moving tip of the shadow.



Interpreting and determining small changes and approximations of certain quantities

Consider the curve $y = f(x)$ on the right. Two points $A(x, y)$ and $B(x + \delta x, y + \delta y)$ are very near to each other on the curve and AT is a tangent to the curve A. Notice that $AC = \delta x$ and $BC = \delta y$.



It is known that the gradient of tangent AT is:

$$\text{The value of } \frac{dy}{dx} \text{ at point } A = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$$

where δy and δx are small changes in y and x respectively.

If δx is very small, that is $\delta x \to 0$, then $\frac{\delta y}{\delta x}$ is the best approximation for $\frac{dy}{dx}$.

So, $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$.

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In general, if δx is a small value, then

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

DISCUSSION

If value of δx is too large, can you use the formula of $\delta y \approx \frac{dy}{dx} \times \delta x$? Explain.

This formula is very useful in finding the approximate change of a quantity caused by a small change in another related quantity. The smaller the value of δx, the more accurate the approximation is. Therefore, we can define that:

For a function $y = f(x)$, where δy is a small change in y and δx is a small change in x,
- When $\delta y > 0$, there is a small increase in y due to a small change in x, that is, δx.
- When $\delta y < 0$, there is a small decrease in y due to a small change in x, that is, δx.

Since $f(x + \delta x) = y + \delta y$ and $\delta y \approx \frac{dy}{dx} \times \delta x$, we will get:

$$f(x + \delta x) \approx y + \frac{dy}{dx}\delta x \text{ or } f(x + \delta x) \approx f(x) + \frac{dy}{dx}\delta x$$

This formula is used to find the approximate value of y.

Example 22

Given $y = x^3$, find
(a) the approximate change in y when x increases from 4 to 4.05,
(b) the approximate change in x when y decreases from 8 to 7.97.

Solution

(a) $y = x^3$

$\frac{dy}{dx} = 3x^2$

When $x = 4$, $\delta x = 4.05 - 4$
$= 0.05$

and $\frac{dy}{dx} = 3(4)^2 = 48$

Then, $\delta y \approx \frac{dy}{dx} \times \delta x$
$= 48 \times 0.05$
$\delta y = 2.4$

Therefore, the approximate change in y, that is δy, is 2.4.
$\delta y > 0$ means there is a small increase in y of 2.4.

(b) When $y = 8$, $x^3 = 8$
$x = 2$
$\delta y = 7.97 - 8 = -0.03$

and $\frac{dy}{dx} = 3(2)^2 = 12$

Then, $\delta y \approx \frac{dy}{dx} \times \delta x$
$-0.03 = 12 \times \delta x$
$\delta x = \frac{-0.03}{12}$
$\delta x = -0.0025$

Therefore, the approximate change in x, that is δx, is -0.0025.
$\delta x < 0$ means there is a small decrease in x of 0.0025.

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Example 23

Given $y = \sqrt{x}$, find

(a) the value of $\frac{dy}{dx}$ when $x = 4$

(b) the approximate value of $\sqrt{4.02}$

Solution

(a) $y = \sqrt{x}$

$= x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1}$

$= \frac{1}{2}x^{-\frac{1}{2}}$

$= \frac{1}{2\sqrt{x}}$

When $x = 4$, $\frac{dy}{dx} = \frac{1}{2\sqrt{4}}$

$= \frac{1}{2(2)}$

$= \frac{1}{4}$

(b) When $x = 4$, $y = \sqrt{4}$

$= 2$

$\delta x = 4.02 - 4$

$= 0.02$

and $\frac{dy}{dx} = \frac{1}{4}$

Using $f(x + \delta x) \approx y + \frac{dy}{dx}\delta x$

$\sqrt{x + \delta x} \approx y + \frac{dy}{dx}\delta x$

$\sqrt{4 + 0.02} = 2 + \frac{1}{4}(0.02)$

$\sqrt{4.02} = 2.005$

Therefore, the approximate value of $\sqrt{4.02}$ is 2.005.

From Example 23, note the table below.

Percentage change in x	Percentage change in y
$\frac{\delta x}{x} \times 100 = \frac{4.02 - 4}{4} \times 100$ $= \frac{0.02}{4} \times 100$ $= 0.5\%$	$\frac{\delta y}{y} \times 100 = \frac{2.005 - 2}{2} \times 100$ $= \frac{0.005}{2} \times 100$ $= 0.25\%$

In general,

If x changes from x to $x + \delta x$, then

- The percentage change in $x = \frac{\delta x}{x} \times 100\%$
- The percentage change in $y = \frac{\delta y}{y} \times 100\%$

Hence, given a function, for example, $y = 3x^2 - 2x - 3$ and x increases by 2% when $x = 2$, can you determine the percentage change in y? Follow Example 24 to solve this kind of problems.

Alternative Method

In Example 23, δy can also be obtained by substitution method.

Given $y = \sqrt{x}$.

When $x = 4$, $y = \sqrt{4}$

$= 2$

When $x = 4.02$, $y = \sqrt{4.02}$

$= 2.005$

So, $\delta y = 2.005 - 2$

$= 0.005$

Hence, $\sqrt{4.02} = y + \delta y$

$= 2 + 0.005$

$= 2.005$

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Example 24

Given $y = 2x^2 - 3x + 4$. When $x = 2$, there is a small change in x by 3%. By using the concept of calculus, find the corresponding percentage change in y.

Solution

Given $y = 2x^2 - 3x + 4$

When $x = 2$, $y = 2(2)^2 - 3(2) + 4$
$= 6$

$\frac{dy}{dx} = 4x - 3$
$= 4(2) - 3$
$= 5$

and $\delta x = \frac{3}{100} \times 2$
$= 0.06$

Then, $\delta y \approx \frac{dy}{dx} \times \delta x$
$= 5 \times 0.06$
$= 0.3$

$\frac{\delta y}{y} \times 100 = \frac{0.3}{6} \times 100$
$= 5$

Thus, the corresponding percentage change in y is 5%.

Self-Exercise 2.14

1. For each of the following functions, find the small corresponding change in y with the given small change in x.
(a) $y = 4x^3 - 3x^2$, when x increases from 1 to 1.05.
(b) $y = 4\sqrt{x} + 3x^2$, when x decreases from 4 to 3.98.

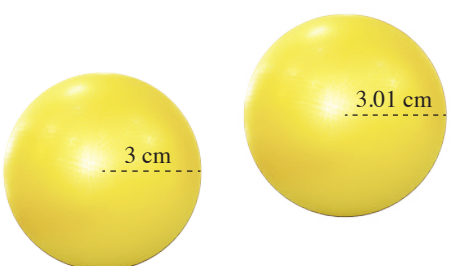
2. For each of the following functions, find the small corresponding change in x with the given small change in y.
(a) $y = 2x^{\frac{3}{2}}$, when y decreases from 16 to 15.7.
(b) $y = \frac{x + 2}{2}$, when y increases from 2 to $2 + p$.

3. Given $y = \frac{16}{x^2}$ find the value of $\frac{dy}{dx}$ when $x = 2$ and determine the approximate value for $\frac{16}{2.02^2}$

4. If $y = x^{\frac{5}{4}}$, find the approximate percentage change in x when there is 4% change in y.

Solving problems involving small changes and approximations of certain quantities

Air is pumped into a spherical ball with a radius of 3 cm. Its radius changes from 3 cm to 3.01 cm. Can you determine the small change in its radius? What about the small change in its volume?



Problems involving small changes can be solved by using the appropriate formula which we have learnt earlier, that is $\delta y \approx \frac{dy}{dx} \times \delta x$.

2.4.8 2.4.9

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Example 25 MATHEMATICAL APPLICATIONS

Find the small change in the volume, V cm^3, of a spherical glass ball when its radius, r cm, increases from 3 to 3.02 cm.

Solution

1 . Understanding the problem

- The radius, r of the glass ball increases from 3 cm to 3.02 cm.
- Find the small change in the volume, V of the glass ball.

2 . Planning the strategy

- Find the value of $\frac{dV}{dr}$ when $r = 3$ cm.
- Use the formula $\delta V \approx \frac{dV}{dr} \times \delta r$.

3 . Implementing the strategy

Let V cm^3 and r cm be the volume and the radius of the glass ball respectively.

Then, $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

When $r = 3$, $\delta r = 3.02 - 3$
$= 0.02$

and $\frac{dV}{dr} = 4\pi(3)^2$
$= 36\pi$

Hence, $\delta V \approx \frac{dV}{dr} \times \delta r$

$= 36\pi \times 0.02$

$\delta V = 2.262$

Therefore, the approximate change in the volume is 2.262 cm^3.

4 . Check and reflect

When $r = 3$ cm,

$V = \frac{4}{3}\pi(3)^3$

$V = 113.0973$ cm^3

When $r = 3.02$ cm,

$V = \frac{4}{3}\pi(3.02)^3$

$V = 115.3744$ cm^3

The change in the volume of the glass ball
$= 115.3744 - 113.0973$
$= 2.277$

Therefore, the approximate change in the volume is 2.277 cm^3.

Self-Exercise 2.15

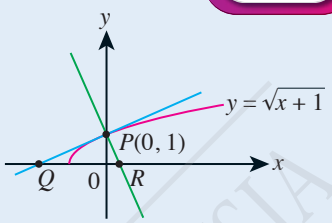
1. The period of oscillation, T second, of a pendulum with a length of l cm is given by $T = 2\pi\sqrt{\frac{l}{10}}$. Find the approximate change in T when l increases from 9 cm to 9.05 cm.
2. The area of a drop of oil which spreads out in a circle increases from 4π cm^2 to 4.01π cm^2. Find the corresponding small change in the radius of the oil.
3. The length of the side of a cube is x cm. Find the small change in the volume of the cube when each side decreases from 2 cm to 1.99 cm.
4. Find the small change in the volume of a sphere when its radius decreases from 5 cm to 4.98 cm.

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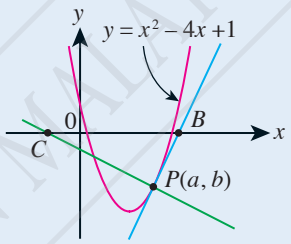
Formative Exercise 2.4

Quiz bit.ly/36yHwhb

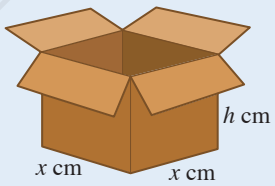
1. The diagram on the right shows a part of the curve $y = \sqrt{x + 1}$. The tangent and the normal to the curve at $P(0, 1)$ intersect the x-axis at Q and R respectively. Find
(a) the equation of the tangent and the coordinates of Q,
(b) the equation of the normal and the coordinates of R,
(c) the area of triangle PQR, in units^2.



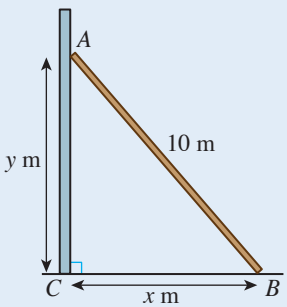
2. The diagram on the right shows the curve $y = x^2 - 4x + 1$ with its tangent and normal at $P(a, b)$. The tangent is perpendicular to the line $2y = 4 - x$ and it meets the x-axis at B. The normal line meets the x-axis at C. Find
(a) the values of a and b,
(b) the equation of the tangent at P and the coordinates of B,
(c) the equation of the normal at P and the coordinates of C,
(d) the area of triangle BPC, in units^2.



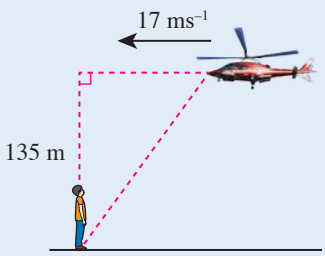
3. The diagram on the right shows an open box with a square base of side x cm and a height of h cm. The box is made from a piece of cardboard with an area of 75 cm^2.
(a) Show that the volume of the box, V cm^3, is given by $V = \frac{1}{4}(75x - x^3)$.
(b) Find the value of x such that the volume, V is maximum and also the maximum volume of the box.



4. The diagram on the right shows a plank AB of length 10 m, leaning on a wall of a building. The end A is y m from the level of the ground and the other end B is x m from the foot of the wall C. Find
(a) the rate of change of end A of the plank if end B slides away from the foot of the wall at a rate of 3 ms^{-1} when $x = 8$ m,
(b) the rate of change of end B of the plank if end A slides down at a rate of 2 ms^{-1} when $y = 6$ m.



5. The diagram on the right shows a helicopter at a height of 135 m from the ground. The helicopter moves horizontally towards the boy at a rate of 17 ms^{-1}. Find the rate of change of the distance between the helicopter and the boy when the horizontal distance between the helicopter and the boy is 72 m.



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REFLECTION CORNER

DIFFERENTIATION

The idea of limits: $\lim\limits_{x \to a} f(x) = L$

Differentiation by first principles

If $y = f(x)$, then $\dfrac{dy}{dx} = \lim\limits_{\delta x \to 0} \dfrac{\delta y}{\delta x}$, where δy is a small change in y and δx is a small change in x.

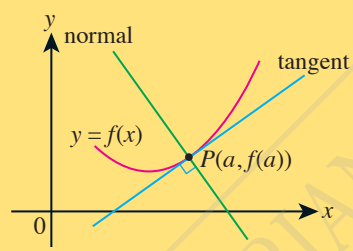
Differentiation formula

- If $y = ax^n$, where a is a constant and n is an integer, then $\dfrac{d}{dx}(ax^n) = anx^{n-1}$.
- If y is a function of u and u is a function of x, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$ (Chain rule)
- If u and v are functions of x, then $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (Product rule)

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$ (Quotient rule)

Applications

Tangent and normal



- Tangent: $y - f(a) = f'(a)(x - a)$
- Normal: $y - f(a) = -\dfrac{1}{f'(a)}(x - a)$

Rates of change of related quantities

If two variables, x and y change with time, t, then

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

Small changes and approximations

If $y = f(x)$ and the small change in x, that is δx, causes a small change in y, that is δy, then

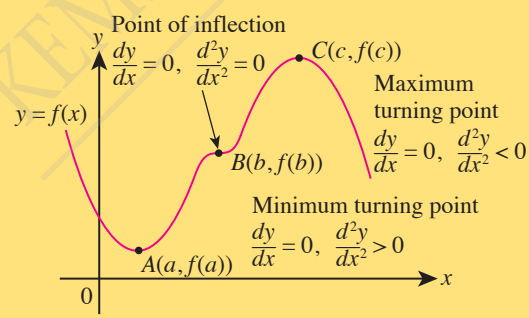
$$\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$$

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

and $f(x + \delta x) \approx y + \delta y$

$$\approx y + \frac{dy}{dx}(\delta x)$$

Stationary points of curve $y = f(x)$



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Journal Writing

1. Compare the method of differentiation used to find the first derivative of a function $y = f(x)$ by using the chain rule, the product rule and the quotient rule.
2. The sketching of tangent test and the second derivatives test are used to determine the nature of turning points. With suitable examples, illustrate the advantages and disadvantages of the two methods.
3. Present four applications of differentiation in a digital folio and exhibit them in front of the class.

Summative Exercise

1. Solve each of the following limits. PL 2

(a) $\lim\limits_{x \to -2} \dfrac{8 + 2x - x^2}{8 - 2x^2}$ (b) $\lim\limits_{x \to 0} \dfrac{\sqrt{1 + x + x^2} - 1}{x}$ (c) $\lim\limits_{x \to k} \dfrac{9 - x^2}{4 - \sqrt{x^2 + 7}} = 8$

2. Given that $\lim\limits_{x \to -1} \dfrac{a - 5}{x + 4} = -3$, find the value of constant a. PL 2

3. Differentiate each of the following with respect to x. PL 2

(a) $\dfrac{1}{2x + 1}$ (b) $4x(2x - 1)^5$ (c) $\dfrac{6}{(2 - x)^2}$ (d) $x\sqrt{x + 3}$

4. Given $y = x(3 - x)$. PL 2

(a) Express $y\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 12$ in terms of x in its simplest form.

(b) Subsequently, find the value of x which satisfies $y\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 12 = 0$.

5. The gradient of the curve $y = ax + \dfrac{b}{x^2}$ at point $\left(-1, -\dfrac{7}{2}\right)$ is 2. Find the values of a and b. PL 3

6. The volume of a sphere increases at a rate of 20π cm^{3}s^{-1}. Find the radius of the sphere when the rate of change of the radius is 0.2 cms^{-1}. PL 2

7. Given $y = \dfrac{14}{\sqrt{6x^3 + 1}}$, find PL 3

(a) the approximate change in y when x increases from 2 to 2.05,
(b) the approximate value of y when $x = 2.05$.

8. Given $y = \dfrac{1}{\sqrt{x}}$, find the approximate percentage change in y when x changes from 4 by 2%. PL 3

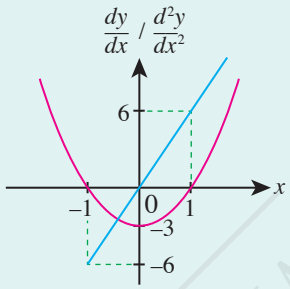
9. Given $y = 3x^2 - 4x + 6$ and there is a small change in x by p% when $x = 2$, find the corresponding percentage change in y. PL 3

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10. The diagram on the right shows graphs $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for function $y = f(x)$. It is given that the function $y = f(x)$ passes through $(-1, 6)$ and $(1, 2)$. Without finding the equation of the function $y = f(x)$, PL 4

(a) determine the coordinates of the maximum and minimum points of the graph function $y = f(x)$,

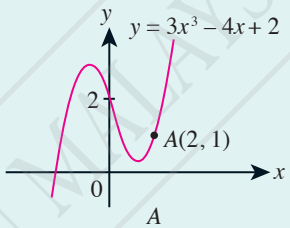
(b) sketch the graph for the function $y = f(x)$.



11. The diagram on the right shows a part of the curve $y = 3x^3 - 4x + 2$. Find PL 3

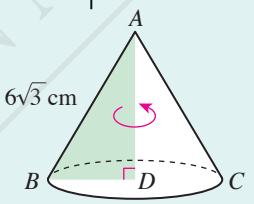
(a) the equation of the tangent at point $A(2, 1)$,

(b) the coordinates of another point on the curve such that the tangent at that point is parallel to the tangent at A.

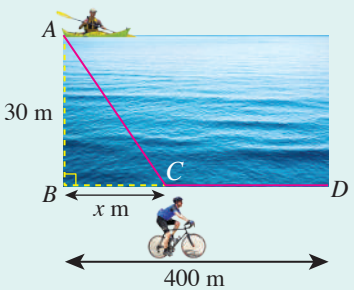


12. In the diagram on the right, ΔADB is a right-angled triangle with a hypotenuse of $6\sqrt{3}$ cm. The triangle is rotated about AD to form a cone ABC. Find PL 4

(a) the height, (b) the volume of the cone, such that the volume generated is maximum.



13. In the diagram on the right, Mukhriz rows his canoe from point A to C where A is 30 m from the nearest point B, which is on the straight shore BD, and C is x m from B. He then cycles from C to D where BD is 400 m. Find the distance from B to C if he rows with a velocity of 40 mmin^{-1} and cycles at 50 mmin^{-1}. PL 5



14. The sides of a cuboid expand at a rate of 2 cms^{-1}. Find the rate of change of the total surface area when its volume is 8 cm^3. PL 3

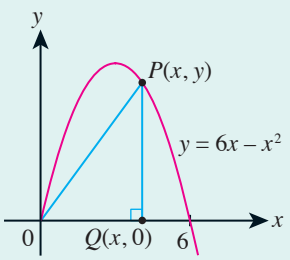
15. The diagram on the right shows a part of the curve $y = 6x - x^2$ which passes through the origin and point $P(x, y)$. PL 3

(a) If Q is point $(x, 0)$, show that the area, A of triangle POQ is given by $A = \frac{1}{2}(6x^2 - x^3)$.

(b) Given that x increases at a rate of 2 units per second, find

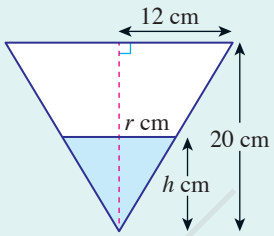
(i) the rate of increase for A when $x = 2$,

(ii) the rate of decrease for A when $x = 5$.



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16. The diagram on the right shows an inverted cone with a base radius of 12 cm and a height of 20 cm. PL 6



(a) If the height of water in the cone is h cm, show that the volume of water, V cm^3, in the cone is $V = \frac{3}{25}\pi h^3$.

(b) Water leaks out through a small hole at the tip of the cone;

(i) find the small change in the volume of water when the height, h decreases from 5 cm to 4.99 cm,

(ii) show that a decrease of $p\%$ in the height of the water will cause a decrease of $3p\%$ in its volume.

MATHEMATICAL EXPLORATION

A multinational beverage company holds a competition to design a suitable container for its new product, a coconut-flavoured drink.

DESIGNING A DRINK CONTAINER COMPETITION

Criteria for the design of the drink container are as follows:

- The capacity of the container is 550 cm^3.
- The shapes of the containers to be considered are cylinders, cone, pyramid, prism, cuboid or cubes. Spherical shape is not allowed.
- Material required to make the tin must be minimum.
- The container must be unique and attractive.



Great prize awaits you!

Join this competition with your classmates. Follow the criteria given and follow the steps given below:

1. Suggest three possible shapes of the containers.
2. For each container with a capacity of 550 cm^3, show the dimensions of the containers with their minimum surface areas. State each minimum surface area.
3. Choose the best design from the three designs to be submitted for the competition by listing down the advantages of the winning design.

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CHAPTER 3

INTEGRATION



What will be learnt?

- Integration as the Inverse of Differentiation
- Indefinite Integral
- Definite Integral
- Application of Integration

List of Learning Standards

bit.ly/2D5bG2c

Have you ever seen an eco-friendly building? The glass panels on the walls allow maximum sunlight to shine in, thus reducing the use of electricity. Do you know that the concept of integration is important in designing the building structure? Engineers apply the knowledge in integration when they design such buildings to ensure that the buildings can withstand strong winds and earthquakes to a certain extent.

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Info Corner

Bonaventura Cavalieri was a well-known Italian mathematician who introduced the concept of integration. He used the concept of indivisibles to find the area under the curve.

In the year 1656, John Wallis from England made significant contribution to the basics of integration by introducing the concept of limits officially.

For more info:

bit.ly/36GUAku

Significance of the Chapter

- In hydrology, engineers use integration in determining the volume of a hydrological system based on the area under a curve with time.
- In civil engineering, integration is used to find the centre of gravity of irregular-shaped objects.
- In the evaluation of car safety, the Head Injury Criterion (HIC) uses integration to assess the extent of head trauma in a collision.

Key words

Differentiation	*Pembezaan*
Integration	*Pengamiran*
Gradient function	*Fungsi kecerunan*
Equation of curve	*Persamaan lengkung*
Indefinite integral	*Kamiran tak tentu*
Definite integral	*Kamiran tentu*
Integration by substitution	*Pengamiran melalui penggantian*
Region	*Rantau*
Volume of revolution	*Isi padu kisaran*

Video on an eco-friendly building

bit.ly/39Oq1vg

3.1 Integration as the Inverse of Differentiation

The photo on the right shows a water tank installed at a factory. Water flows out of the tank at a rate given by $\frac{dV}{dt} = 5t + 2$, where V is the volume, in m^3, and t is the time, in hour. The tank will be emptied within 5 hours.

With this rate of water used from the tank, can we determine the volume of water in the tank at a certain time?

The relation between differentiation and integration

We have learnt how to differentiate a given function $y = f(x)$. Consider the function $y = 3x^2 + 4x + 5$, then we get $\frac{dy}{dx} = 6x + 4$.

Integration is a process which is quite similar to differentiation but it is denoted by the symbol $\int \ldots dx$. What is the relation between differentiation and integration? Let's explore further.

Recall

- If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$.
- If $y = a$, then $\frac{dy}{dx} = 0$.
- If $y = ax$, then $\frac{dy}{dx} = a$.

Discovery Activity 1

Pair | 21st cl | STEM | CT

Aim: To determine the relation between differentiation and integration

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Click on the functions given and observe the graphs of each of them.
3. With your partner, discuss:
   (a) the relation between the graphs of function $f(x)$, $f'(x)$ and $g(x)$,
   (b) the relation between the graphs of function $h(x)$, $h'(x)$ and $k(x)$,
   (c) the relation between the graphs of function $m(x)$, $m'(x)$ and $n(x)$.
4. Then, present your findings to the class.
5. Members from other pairs can ask questions.

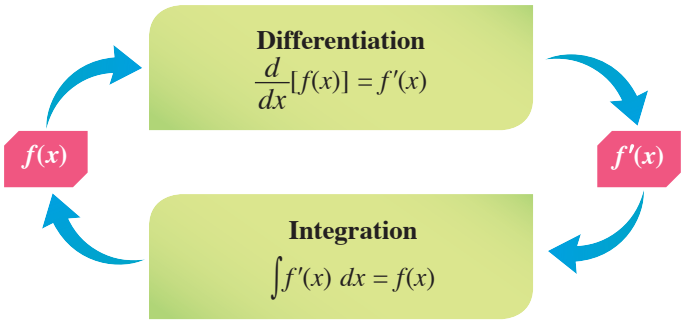
ggbm.at/ccdbhvpd

From Discovery Activity 1, it is found:

- The graph of function $g(x) = \int f'(x)\,dx$ is the same as the graph of function $f(x)$.
- The graph of function $k(x) = \int h'(x)\,dx$ is the same as the graph of function $h(x)$.
- The graph of function $n(x) = \int m'(x)\,dx$ is the same as the graph of function $m(x)$.

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Hence, we can conclude that integration is in fact the **reverse process** of differentiation. The functions $f(x)$, $h(x)$ and $m(x)$ are known as **antiderivatives** of functions $g(x)$, $k(x)$ and $n(x)$ respectively.



In general,

If $\frac{d}{dx}[f(x)] = f'(x)$, then the integral of $f'(x)$ with respect to x is $\int f'(x)\,dx = f(x)$.

HISTORY GALLERY



Gottfried Wilhelm Leibniz, a German mathematician, was the one who introduced the integral symbol $\int$ in 1675. He adapted it from the alphabet ſ or long s.

CHAPTER 3

Example 1

Given $\frac{d}{dx}(4x^2) = 8x$, find $\int 8x\,dx$.

Solution

Differentiation of $4x^2$ is $8x$.
By the reverse of differentiation, the integration of $8x$ is $4x^2$.
Hence, $\int 8x\,dx = 4x^2$.

Flash Quiz

Give three examples in daily lives that can illustrate that integration is the reverse of differentiation.

Example 2

The coal production from a coal mine is given by $K = 48\,000t - 100t^3$, where K is the mass of coal produced, in tonnes, and t is the time, in years.

(a) Find the rate of production of coal, $\frac{dK}{dt}$, in terms of t.

(b) If the rate of production of coal is given by $\frac{dK}{dt} = 96\,000 - 600t^2$, find the mass of coal produced, in tonnes, in the fourth year.



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Solution

(a) Given $K = 48\,000t - 100t^3$.

Then, $\frac{dK}{dt} = 48\,000 - 300t^2$.

(b) Given $\frac{dK}{dt} = 96\,000 - 600t^2$
$= 2(48\,000 - 300t^2)$

By the reverse of differentiation, the integration of $48\,000 - 300t^2$ is $48\,000t - 100t^3$.

Hence, $\int 2(48\,000 - 300t^2)\,dt = 2(48\,000t - 100t^3)$
$= 96\,000t - 200t^3$

Therefore, the mass of coal produced in the fourth year $= 96\,000(4) - 200(4)^3$
$= 371\,200$ tonnes

Self-Exercise 3.1

1. Given $\frac{d}{dx}(5x^3 + 4x) = 15x^2 + 4$, find $\int (15x^2 + 4)\,dx$.

2. Given $\frac{d}{dx}(8x^3) = 24x^2$, find $\int 24x^2\,dx$.

3. The usage of water at mall A is given by the function $J = 100t^3 + 30t^2$, where J is the volume of water used, in litres, and t is the time, in days.

(a) Find the rate of water used at mall A, in terms of t.

(b) If the rate of usage of the water in mall A changes according to $\frac{dJ}{dt} = 1\,500t^2 + 300t$, find the volume, in litres, used on the second day.

Formative Exercise 3.1

Quiz bit.ly/2R1cQP7

1. Given $y = 3(2x + 2)^3$, find $\frac{dy}{dx}$. Subsequently, find $\int [18(2x + 2)^2]\,dx$.

2. Given $f(x) = \frac{5x + 2}{2 - 3x}$, find $f'(x)$ and $\int f'(x)\,dx$.

3. Given $y = 5(x + 2)^3$ and $\frac{dy}{dx} = h(x + 2)^k$, find the value of $h + k$. Subsequently, find the value of $\frac{1}{10}\int \left(\frac{dy}{dx}\right) dx$ where $x = 2$.

4. Given $f(x) = 3x(2x + 1)^2$ and $\int (12x^2 + 8x + 1)\,dx = af(x)$, find the value of a.

5. The profit function from the sale of bus tickets of company K is given by $A = 100t^2 + 50t^3$, where A is the profit obtained, in RM, and t is the time, in days.

(a) Find the rate of profit obtained by the bus company after 5 days.

(b) Given that the rate of profit obtained from another bus company H is given by $\frac{dA}{dt} = 30t^2 + 40t$, which company gets more profit on the 10th day?

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3.2 Indefinite Integral

The photo shows a Young Doctors' club in a school taking the blood pressure of their peers. What method is used to determine the blood pressure in the aorta, after t seconds for a normal person?

By applying the **indefinite integral** to the rate of blood pressure, we can determine the blood pressure of a person.

Indefinite integral formula

Discovery Activity 2 Pair 21st cl

Aim: To derive the formula for indefinite integral by induction

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Complete the table for Case 1, taking turns with your friend.
3. Based on your table, derive the formula for indefinite integral by induction.
4. Repeat steps 2 and 3 for Case 2.
5. Exhibit your friend's and your work in the class.
6. Go around and observe the findings from other groups.

bit.ly/2FzEXQ5

From Discovery Activity 2 results, we found that:

- For a constant a,

$$\int a\,dx = ax + c, \text{ where } a \text{ and } c \text{ are constants.}$$

- For a function ax^n,

$$\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c, \text{ where } a \text{ and } c \text{ are constants, } n \text{ is an integer and } n \neq -1.$$

Excellent Tip

Steps to find the integral of ax^n with respect to x, where a is a constant, n is an integer and $n \neq -1$:

1. Add 1 to the index of x.
2. Divide the term with the new index.
3. Add the constant c with the integrals.

In general, the function $ax + c$ and $\frac{ax^{n+1}}{n+1} + c$ are known as **indefinite integrals** for constant a with respect to x and function ax^n with respect to x respectively.

Consider the following cases.

Case 1

$y = 5x$, $\frac{dy}{dx} = 5$ and $\int 5\,dx = 5x$

Case 2

$y = 5x + 2$, $\frac{dy}{dx} = 5$ and $\int 5\,dx = 5x + 2$

Case 3

$y = 5x - 3$, $\frac{dy}{dx} = 5$ and $\int 5\,dx = 5x - 3$

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Notice that differentiating those three cases give the same value of $\frac{dy}{dx}$, even though each of them has a different constant. This constant is known as the **constant of integration** and represented by the symbol c. The constant c is added as a part of indefinite integral for a function. For example, $\int 5\ dx = 5x + c$.

Indefinite integral for algebraic functions

The indefinite integral formula can be used to find indefinite integral of a constant or an algebraic function.

Example 3

Integrate each of the following with respect to x.

(a) 12 (b) $\frac{1}{2}$ (c) -0.5

Solution

(a) $\int 12\ dx = 12x + c$

(b) $\int \frac{1}{2}\ dx = \frac{1}{2}x + c$

(c) $\int -0.5\ dx = -0.5x + c$

Example 4

Find the indefinite integral for each of the following.

(a) $\int x^3\ dx$ (b) $\int \frac{2}{x^2}\ dx$

Solution

(a) $\int x^3\ dx = \frac{x^{3+1}}{3+1} + c$
$= \frac{x^4}{4} + c$

(b) $\int \frac{2}{x^2}\ dx = 2\int x^{-2}\ dx$
$= 2\left(\frac{x^{-2+1}}{-2+1}\right) + c$
$= -2x^{-1} + c$
$= -\frac{2}{x} + c$

Excellent Tip

$\int ax^n\ dx = a\int x^n\ dx$

Flash Quiz

Find the integral for each of the following.

(a) $\int dx$

(b) $\int 0\ dx$

(c) $\int |x|\ dx$

In the chapter on differentiation, we have learnt the method of differentiating a function such as $h(x) = 3x^2 + 5x$, by expressing $f(x) = 3x^2$ and $g(x) = 5x$.

A similar approach will be used to find the integral for functions with addition or subtraction of algebraic terms.

If $f(x)$ and $g(x)$ are functions, then

$$\int [f(x) \pm g(x)]\ dx = \int f(x)\ dx \pm \int g(x)\ dx$$

Information Corner

$\int [f(x) \pm g(x)]\ dx$
$= \int f(x)\ dx \pm \int g(x)\ dx$
is also known as addition or subtraction rule.

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Example 5

Find the integral for each of the following.

(a) $\int (3x^2 + 2)\,dx$ (b) $\int (x - 2)(x + 6)\,dx$ (c) $\int x^2\left(3 + \frac{1}{x^5}\right)dx$

Solution

(a) $\int (3x^2 + 2)\,dx$

$= \int 3x^2\,dx + \int 2\,dx$

$= \frac{3x^3}{3} + 2x + c$

$= x^3 + 2x + c$

(b) $\int (x - 2)(x + 6)\,dx$

$= \int (x^2 + 4x - 12)\,dx$

$= \int x^2\,dx + \int 4x\,dx - \int 12\,dx$

$= \frac{x^3}{3} + \frac{4x^2}{2} - 12x + c$

$= \frac{x^3}{3} + 2x^2 - 12x + c$

(c) $\int x^2\left(3 + \frac{1}{x^5}\right)dx = \int \left(3x^2 + \frac{1}{x^3}\right)dx$

$= \int \left(3x^2 + x^{-3}\right)dx$

$= \int 3x^2\,dx + \int x^{-3}\,dx$

$= \frac{3x^3}{3} + \frac{x^{-2}}{-2} + c$

$= x^3 - \frac{1}{2x^2} + c$

DISCUSSION

Integration of functions containing algebraic terms added or subtracted together will have only one constant of integration. Discuss.

Self-Exercise 3.2

1. Find the indefinite integral for each of the following.

(a) $\int 2\,dx$ (b) $\int \frac{5}{6}\,dx$ (c) $\int -2\,dx$ (d) $\int \frac{\pi}{3}\,dx$

2. Integrate each of the following with respect to x.

(a) $3x^2$ (b) $\frac{4}{3}x^3$ (c) $-x$ (d) $-\frac{2}{x^2}$

(e) $\frac{3}{x^3}$ (f) $3\sqrt{x}$ (g) $\frac{2}{\sqrt[3]{x}}$ (h) $\left(-\frac{3}{\sqrt{x}}\right)^3$

3. Integrate each of the following with respect to x.

(a) $2x + 3$ (b) $4x^2 + 5x$ (c) $\frac{1}{2}x^3 + 5x - 2$ (d) $\frac{3}{x^2} + 4x - 2$

4. Find the indefinite integral for each of the following.

(a) $\int (x + 2)(x - 4)\,dx$ (b) $\int x^2(3x^2 + 5x)\,dx$ (c) $\int \left(5x^2 - 3\sqrt{x}\right)dx$

(d) $\int (5x - 3)^2\,dx$ (e) $\int \left(\frac{5x^2 - 3x}{x}\right)dx$ (f) $\int \left(x + \sqrt{x}\right)^2 dx$

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Indefinite integral for functions in the form of $(ax + b)^n$, where a and b are constants, n is an integer and $n \neq -1$

Earlier we have studied how to integrate the function such as $y = 2x + 1$. How do we find the integral for the function $y = (2x + 1)^8$?

The expression of $(2x + 1)^8$ is difficult to expand. Hence, functions like this will require us to use **substitution method**.

Let's consider the function $y = \int (ax + b)^n \, dx$, where a and b are constants, n is an integer and $n \neq -1$, and thus, $\frac{dy}{dx} = (ax + b)^n$.

Let $u = ax + b$

Then, $\frac{du}{dx} = a$

and $\frac{dy}{dx} = u^n$

With chain rule,

$$\frac{dy}{du} = \frac{dy}{dx} \times \frac{dx}{du}$$
$$= \frac{dy}{dx} \times \frac{1}{\left(\frac{du}{dx}\right)}$$

Substitute $\frac{dy}{dx} = u^n$ and $\frac{du}{dx} = a$, and we get

$$\frac{dy}{du} = u^n \times \frac{1}{a}$$
$$y = \int \frac{u^n}{a} \, du$$
$$\int (ax + b)^n \, dx = \int \frac{u^n}{a} \, du$$
$$= \frac{1}{a} \int u^n \, du$$
$$= \frac{1}{a} \left[\frac{u^{n+1}}{n + 1}\right] + c$$

Substitute $u = ax + b$, and we get

$$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$$

Thus,

$\int (ax + b)^n \, dx = \frac{(ax + b)^{n+1}}{a(n + 1)} + c$, where a and b are constants, n is an integer and $n \neq -1$.

Recall

For a function $y = g(u)$ and $u = h(x)$,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Information Corner

The expression $(ax + b)^n$ can be expanded by using binomial theorem. The general binomial theorem formula for the expression $(ax + b)^n$ is $\sum_{k=0}^{n} [{}^nC_k (ax)^{n-k} (b)^k]$, where k and n are integers and a and b are constants.

DISCUSSION

Using the formula on the left, can you find the integral of $\int (3x^2 + 3)^3 \, dx$?

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Example 6

By using substitution method, find the indefinite integral for each of the following.

(a) $\int (3x + 5)^5 \, dx$

(b) $\int \sqrt{5x + 2} \, dx$

Solution

(a) Let $u = 3x + 5$

Then, $\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$$\int (3x + 5)^5 \, dx = \int \frac{u^5}{3} \, du$$
$$= \frac{1}{3}\left(\frac{u^6}{6}\right) + c$$
$$= \frac{(3x + 5)^6}{18} + c$$

(b) Let $u = 5x + 2$

Then, $\frac{du}{dx} = 5$

$dx = \frac{du}{5}$

$$\int \sqrt{5x + 2} \, dx = \int \frac{\sqrt{u}}{5} \, du$$
$$= \int \frac{u^{\frac{1}{2}}}{5} \, du$$
$$= \frac{2}{15} u^{\frac{3}{2}} + c$$
$$= \frac{2}{15}(5x + 2)^{\frac{3}{2}} + c$$

Example 7

Integrate each the following with respect to x.

(a) $(2 - 3x)^4$

(b) $\frac{3}{(5x - 3)^6}$

Solution

(a) $$\int (2 - 3x)^4 \, dx = \frac{(2 - 3x)^5}{-3(5)} + c$$
$$= -\frac{(2 - 3x)^5}{15} + c$$

(b) $$\int \frac{3}{(5x - 3)^6} \, dx = \int 3(5x - 3)^{-6} \, dx$$
$$= \frac{3(5x - 3)^{-5}}{5(-5)} + c$$
$$= -\frac{3}{25(5x - 3)^5} + c$$

Self-Exercise 3.3

1. Find the indefinite integral for each of the following by using substitution method.

(a) $\int (x - 3)^2 \, dx$

(b) $\int (3x - 5)^9 \, dx$

(c) $\int 4(5x - 2)^5 \, dx$

(d) $\int \frac{(7x - 3)^4}{3} \, dx$

(e) $\int \frac{12}{(2x - 6)^3} \, dx$

(f) $\int \frac{2}{3(3x - 2)^2} \, dx$

2. Integrate each of the following with respect to x.

(a) $(4x + 5)^4$

(b) $2(3x - 2)^3$

(c) $(5x - 11)^4$

(d) $\frac{(3x - 2)^5}{5}$

(e) $\frac{5}{(6x - 3)^6}$

(f) $\frac{12}{(3x - 5)^8}$

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Equation of a curve from its gradient function

The constant of integration, c can be determined by substituting the given value of x with its corresponding value of y into the result of integration of the gradient function.

Example 8

Determine the constant of integration, c for $\frac{dy}{dx} = 4x^3 + 6x^2 - 3$ where $y = 25$ when $x = 2$.

Solution

Given $\frac{dy}{dx} = 4x^3 + 6x^2 - 3$.

Then, $y = \int (4x^3 + 6x^2 - 3)\, dx$

$$y = \frac{4x^4}{4} + \frac{6x^3}{3} - 3x + c$$

$$y = x^4 + 2x^3 - 3x + c$$

When $x = 2$ and $y = 25$,

$$25 = 2^4 + 2(2)^3 - 3(2) + c$$

$$c = -1$$

Thus, the constant of integration, c for $\frac{dy}{dx} = 4x^3 + 6x^2 - 3$ is -1.

The gradient function, $\frac{dy}{dx}$ or $f'(x)$ of a curve can be obtained from the equation of the curve $y = f(x)$ by differentiation. Conversely, the equation of the curve can be obtained from the gradient function by integration. In general,

> Given the gradient function $\frac{dy}{dx} = f'(x)$, the equation of curve for that function is $y = \int f'(x)\, dx$.

Example 9

The gradient function of a curve at point (x, y) is given by $\frac{dy}{dx} = 15x^2 + 4x - 3$.

(a) If the curve passes through the point $(-1, 2)$, find the equation of the curve.

(b) Subsequently, find the value of y when $x = 1$.

Solution

(a) Given $\frac{dy}{dx} = 15x^2 + 4x - 3$.

Then, $y = \int (15x^2 + 4x - 3)\, dx$

$$y = 5x^3 + 2x^2 - 3x + c$$

When $x = -1$ and $y = 2$,

$$2 = 5(-1)^3 + 2(-1)^2 - 3(-1) + c$$

$$c = 2$$

Thus, the equation of the curve is $y = 5x^3 + 2x^2 - 3x + 2$.

(b) When $x = 1$,

$$y = 5(1)^3 + 2(1)^2 - 3(1) + 2$$

$$y = 6$$

Then, $y = 6$ when $x = 1$.

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Self-Exercise 3.4

1. Find the constant of integration, c for the following gradient functions.

(a) $\frac{dy}{dx} = 4x - 2$, $y = 7$ when $x = -1$

(b) $\frac{dy}{dx} = -6x - \frac{6}{x^3}$, $y = 6$ when $x = -1$

2. Given $\frac{dy}{dx} = 20x^3 - 6x^2 - 6$ and $y = 2$ when $x = 1$, find the value of y when $x = \frac{1}{2}$.

3. Find the equation of curve for each gradient function which passes through the given point.

(a) $\frac{dy}{dx} = 9x^2 - 2$, at point $(1, 6)$

(b) $\frac{dy}{dx} = 10x - 2$, at point $(2, 13)$

(c) $\frac{dy}{dx} = 24x^2 - 5$, at point $(1, 1)$

(d) $\frac{dy}{dx} = 18x^2 + 10x$, at point $(-2, -10)$

CHAPTER 3

Formative Exercise 3.2

Quiz bit.ly/2R2JnUX

1. Find the indefinite integral for each of the following.

(a) $\int \frac{1}{2}\, dx$

(b) $\int \frac{5}{3x^3}\, dx$

(c) $\int \frac{1}{\sqrt{x}}\, dx$

(d) $\int \left(\frac{2}{x^3} - \frac{3}{x^4}\right) dx$

2. Integrate each of the following with respect to x.

(a) $\frac{5x^2 - 3x^3}{x}$

(b) $\frac{6x^3 + 2x^2}{2x^2}$

(c) $(5 - 6x)^3$

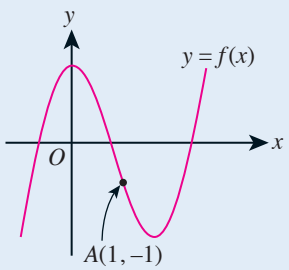
(d) $\frac{1}{\sqrt[4]{5 - 2x}}$

3. It is given that $\frac{dy}{dx} = 10x + \frac{p}{x^2}$, where p is a constant. If $\frac{dy}{dx} = 20\frac{1}{2}$ and $y = 19$ when $x = 2$, find the value of p. Subsequently, find the value of y when $x = -2$.

4. (a) Given $\frac{dy}{dx} = 4x^3 - 15x^2 + 6$ and $y = -20$ when $x = 3$, find the value of y when $x = -2$.

(b) Given $\frac{dy}{dx} = 2x + 2$ and $y = 2$ when $x = 2$, find the values of x when $y = -6$.

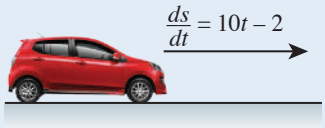
5. The diagram on the right shows a curve that passes through point $A(1, -1)$. Given the gradient function of that curve is $\frac{dy}{dx} = 3x^2 - 8x$, find the equation for that curve.



6. It is given that the gradient of a normal to a curve at one point is $\frac{1}{6x - 2}$. If the curve passes through point $(2, 2)$, find the equation for that curve.

7. It is given that the gradient function of a curve is $ax + b$. The gradient of the curve at $(-2, 8)$ is -7 and the gradient of the curve at $(0, 6)$ is 5. Find the values of a and b. Then, find the equation of the curve.

8. The diagram on the right shows a car being driven on a straight road. It is given that the rate of change of the displacement function of the car is $\frac{ds}{dt} = 10t - 2$ and $s = 8$ m when $t = 1$ s. Find the displacement, in m, when $t = 3$ s.



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3.3 Definite Integral

The Bakun Hydroelectric Dam in Sarawak is the largest hydroelectric power station in Malaysia. How can the engineers building the dams ensure that the dams are built with good safety features?

By applying definite integrals, the engineers were able to determine the surface area and the volume of water kept in the reservoir region. This allowed them to determine the thickness of the walls of the dams that were built to withstand the water pressure in the reservoir.

Value of definite integral for algebraic functions

You have already learnt that the indefinite integral of a function $f(x)$ with respect to x is $\int f(x)\,dx = g(x) + c$, where $g(x)$ is a function of x and c is a constant.

The definite integral of a function $f(x)$ with respect to x with the interval from $x = a$ to $x = b$ can be written as:

$$\begin{aligned}\int_a^b f(x)\,dx &= \big[g(x) + c\big]_a^b \\ &= [g(b) + c] - [g(a) + c] \\ &= g(b) - g(a)\end{aligned}$$

Example 10

Find the value for each of the following.

(a) $\int_2^3 x^2\,dx$ (b) $\int_{-1}^4 (3x^2 + 2x)\,dx$

Solution

(a) $\int_2^3 x^2\,dx$

$= \left[\frac{x^3}{3}\right]_2^3$

$= \frac{3^3}{3} - \frac{2^3}{3}$

$= \frac{19}{3}$

(b) $\int_{-1}^4 (3x^2 + 2x)\,dx$

$= \left[\frac{3x^3}{3} + \frac{2x^2}{2}\right]_{-1}^4$

$= \left[x^3 + x^2\right]_{-1}^4$

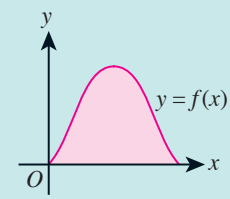
$= [4^3 + 4^2] - [(-1)^3 + (-1)^2]$

$= 80$

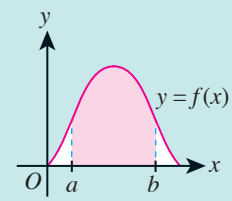
Information Corner

The area under the curve can be determined by integrating the curve function. For a function $y = f(x)$:

(a) Indefinite integral, $\int f(x)\,dx$



(b) Definite integral, $\int_a^b f(x)\,dx$



Flash Quiz

Find the value of

(a) $\int_1^2 1\,dx$

(b) $\int_1^2 0\,dx$

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Example 11

Find the value for each of the following.

(a) $\int_1^2 \left(\frac{x^3 - 2x^2}{x^2}\right) dx$

(b) $\int_2^4 (2x - 5)^4 \, dx$

Solution

(a) $\int_1^2 \left(\frac{x^3 - 2x^2}{x^2}\right) dx$

$= \int_1^2 \left(\frac{x^3}{x^2} - \frac{2x^2}{x^2}\right) dx$

$= \int_1^2 (x - 2) \, dx$

$= \left[\frac{x^2}{2} - 2x\right]_1^2$

$= \left[\frac{2^2}{2} - 2(2)\right] - \left[\frac{1^2}{2} - 2(1)\right]$

$= -2 - \left(-\frac{3}{2}\right)$

$= -\frac{1}{2}$

(b) $\int_2^4 (2x - 5)^4 \, dx$

$= \left[\frac{(2x - 5)^5}{2(5)}\right]_2^4$

$= \left[\frac{(2(4) - 5)^5}{10}\right] - \left[\frac{(2(2) - 5)^5}{10}\right]$

$= \frac{243}{10} - \left(-\frac{1}{10}\right)$

$= \frac{122}{5}$

CHAPTER 3

What are the characteristics of a definite integral? To know more, let's carry out the following exploration.

Discovery Activity 3

Group | 21st cl | STEM | CT

Aim: To determine the characteristics of a definite integral

ggbm.at/j3yzvngv

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Click on all the boxes to see the regions for each definite integral.
3. Observe the regions formed and record the value of each definite integral on a piece of paper.
4. Then, map each of the following expressions on the left to a correct expression on the right.

$\int_2^2 3x^2 \, dx$	$\int_1^6 3x^2 \, dx$
$\int_2^6 3x^2 \, dx$	$3\int_2^6 3x^2 \, dx$
$\int_2^6 3(3x^2) \, dx$	$\int_2^6 3x^2 \, dx + \int_2^6 6x \, dx$
$\int_1^4 3x^2 \, dx + \int_4^6 3x^2 \, dx$	$-\int_6^2 3x^2 \, dx$
$\int_2^6 (3x^2 + 6x) \, dx$	0

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5. Draw a general conclusion deductively from each of the results obtained.
6. Each group appoints a representative to present the findings to the class.
7. Members from other groups are encouraged to ask questions on the findings.

From Discovery Activity 3, the characteristics of definite integrals are as follows:

For the functions $f(x)$ and $g(x)$,

(a) $\int_a^a f(x)\,dx = 0$

(b) $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

(c) $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$, where k is a constant

(d) $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$, where $a < b < c$

(e) $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

Example 12

Given $\int_1^3 f(x)\,dx = 4$, $\int_3^5 f(x)\,dx = 3$ and $\int_1^3 g(x)\,dx = 12$, find

(a) $\int_3^1 f(x)\,dx$ (b) $\int_1^3 [f(x) + g(x)]\,dx$ (c) $\int_1^5 f(x)\,dx$

Solution

(a) $\int_3^1 f(x)\,dx$
$= -\int_1^3 f(x)\,dx$
$= -4$

(b) $\int_1^3 [f(x) + g(x)]\,dx$
$= \int_1^3 f(x)\,dx + \int_1^3 g(x)\,dx$
$= 4 + 12$
$= 16$

(c) $\int_1^5 f(x)\,dx$
$= \int_1^3 f(x)\,dx + \int_3^5 f(x)\,dx$
$= 4 + 3$
$= 7$

Example 13

Given $\int_2^5 f(x)\,dx = 12$, find the value of h if $\int_2^5 [hf(x) - 3]\,dx = 51$.

Solution

$\int_2^5 [hf(x) - 3]\,dx = 51$

$h\int_2^5 f(x)\,dx - \int_2^5 3\,dx = 51$

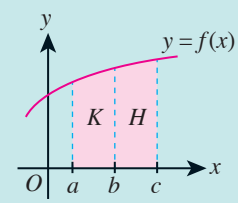
$12h - \left[3x\right]_2^5 = 51$

$12h - [3(5) - 3(2)] = 51$

$12h - 9 = 51$

$h = 5$

Information Corner



The total area of the shaded region = Area of the region K + Area of the region H

$\int_a^c f(x)\,dx$
$= \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$

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Self-Exercise 3.5

1. Find the value for each of the following.

(a) $\int_2^4 x^3\,dx$ (b) $\int_1^4 \frac{2}{x^2}\,dx$ (c) $\int_1^5 (2x^2 + 3x)\,dx$

(d) $\int_2^6 \left(\frac{1}{x^3} - 2x\right) dx$ (e) $\int_1^3 \left(3x - \sqrt{x}\right) dx$ (f) $\int_3^5 \left(x - \frac{1}{\sqrt{x}}\right) dx$

2. Find the value for each of the following definite integrals.

(a) $\int_2^4 \left(\frac{x^3 + x^2}{x}\right) dx$ (b) $\int_1^3 \left(\frac{5 + x^2}{x^2}\right) dx$ (c) $\int_1^5 \left(\frac{(2x + 3)(x - 2)}{x^4}\right) dx$

(d) $\int_3^4 (3x - 4)^2\,dx$ (e) $\int_{-3}^{-1} \frac{3}{(5 - 3x)^3}\,dx$ (f) $\int_{-2}^{0} \frac{2}{\sqrt{3 - 2x}}\,dx$

3. Given $\int_2^5 f(x)\,dx = 3$, find the value for each of the following.

(a) $\int_5^2 f(x)\,dx$ (b) $\int_2^5 \frac{1}{2} f(x)\,dx$ (c) $\int_2^5 [3f(x) - 2]\,dx$

4. Given $\int_3^7 f(x)\,dx = 5$ and $\int_3^7 k(x)\,dx = 7$, find the value for each of the following.

(a) $\int_3^7 [f(x) + k(x)]\,dx$ (b) $\int_3^5 f(x)\,dx - \int_7^5 f(x)\,dx$ (c) $\int_3^7 [f(x) + 2x]\,dx$

The relation between the limit of the sum of areas of rectangles and the area under a curve

Discovery Activity 4

Group 21st cl STEM CT

Aim: To investigate the relation between the limit of the sum of areas of rectangles and the area under a curve

ggbm.at/cnpjf9hd

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Let n be the number of rectangles under the curve $y = -x^2 + 6x$.
3. Drag the cursor n from left to right. Notice the area of the region under the curve $y = -x^2 + 6x$ as n changes.
4. Then, copy and complete the table below.

Number of rectangles, n	Sum of the areas of the rectangles under the curve	Area of the region under the actual curve
1		
2		
⋮	⋮	
20		

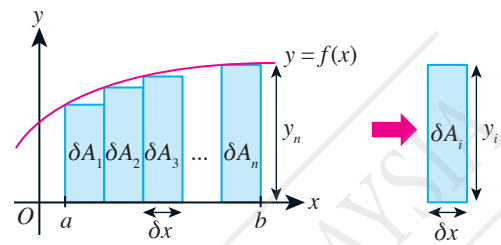
5. Together with your group members, discuss the relation between the sum of the areas of the rectangles under the curve with the area under the actual curve.
6. Present your findings to the class.

3.3.1 3.3.2

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From Discovery Activity 4, it is found that as the number of rectangles under the curve $y = f(x)$ increases, the sum of the areas of the rectangles under the curve approaches the actual area under the curve.

Look at the diagram of the curve $y = f(x)$. The area under the curve $y = f(x)$ from $x = a$ to $x = b$ can be divided into n thin rectangular vertical strips. As the number of strips increases, the width of each rectangle becomes narrower.



The width of each rectangular strip can be written as δx, where $\delta x = \frac{b - a}{n}$.

It is found that:

Area of each rectangular strip, $\delta A_i \approx$ Length of the rectangular strip
$\times$ Width of the rectangular strip
$\approx y_i \times \delta x$
$\approx y_i \delta x$

Area of n rectangular strips $\approx \delta A_1 + \delta A_2 + \delta A_3 + \ldots + \delta A_n$

$$\approx \sum_{i=1}^{n} \delta A_i$$

$$\approx \sum_{i=1}^{n} y_i \delta x$$

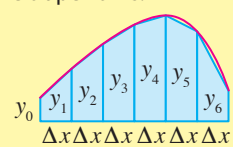
As the number of strips becomes sufficiently large, that is $n \to \infty$, then $\delta x \to 0$.

In general,

$$\text{Area under the curve} = \lim_{\delta x \to 0} \sum_{i=1}^{n} y_i \delta x = \int_a^b y \, dx$$

DISCUSSION

The area under the curve can be related to the limit of the sum of the areas of the trapeziums.



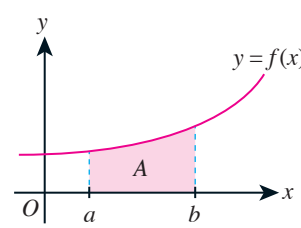
Based on the relation above, construct the formula for $\int_a^b f(x)\,dx$.

Area of a region

Area of a region between the curve and the x-axis

The diagram on the right shows a region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The formula for the area of the region A is given by:

$$A = \int_a^b y \, dx$$



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Discovery Activity 5

Group 21st cl STEM CT

Aim: To determine the area above and below the x-axis

Steps:

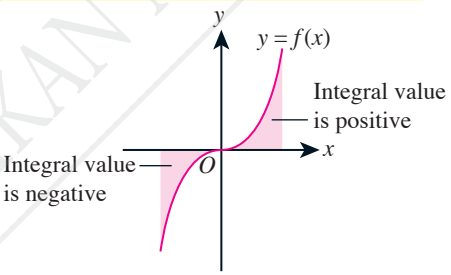
1. Scan the QR code on the right or visit the link below it.

bit.ly/3iEXP1M

2. Observe the area under the curve $y = \frac{1}{3}x^3$ which is displayed on the plane.
3. Drag point a to $x = 0$ and point b to $x = 5$.
4. Take note of the location of the area formed with its corresponding sign of the value of the area.
5. Repeat steps 3 and 4 by changing point a to $x = -5$ and point b to $x = 0$.
6. Record the values of the following definite integrals with their corresponding locations of the area.
   (a) $\int_0^5 \frac{1}{3}x^3\,dx$ (b) $\int_{-5}^0 \frac{1}{3}x^3\,dx$
7. Discuss your group's findings to the class.

From Discovery Activity 5, we obtained that:

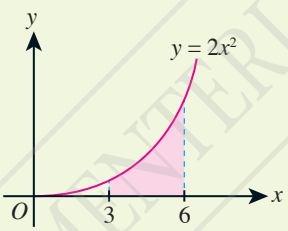
For the value of the area bounded by the curve and the x-axis,
- If the region is below the x-axis, then the integral value is **negative**.
- If the region is above the x-axis, then the integral value is **positive**.
- The areas of both regions are positive.



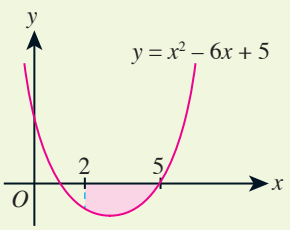
Example 14

Find the area for each of the following shaded regions.

(a)



(b)



Solution

(a) Area of the region $= \int_3^6 y\,dx$

$= \int_3^6 2x^2\,dx$

$= \left[\frac{2x^3}{3}\right]_3^6$

$= \frac{2(6)^3}{3} - \frac{2(3)^3}{3}$

$= 126$

Hence, the area of the shaded region is 126 units^2.

Technology Click

Use Photomath application to find the integral of a function.

bit.ly/2QNZ3LJ

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(b) Area of the region

$= \int_2^5 y\,dx$

$= \int_2^5 (x^2 - 6x + 5)\,dx$

$= \left[\frac{x^3}{3} - \frac{6x^2}{2} + 5x\right]_2^5$

$= \left[\frac{5^3}{3} - \frac{6(5)^2}{2} + 5(5)\right] - \left[\frac{2^3}{3} - \frac{6(2)^2}{2} + 5(2)\right]$

$= -9$

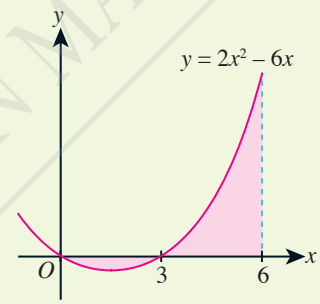
Hence, the area of the region is 9 units^2.

Information Corner

The negative sign of the value of the definite integral is only used to indicate that the area is under the x-axis. Hence, the negative sign can be ignored for its area.

Example 15

The diagram on the right shows a part of the curve $y = 2x^2 - 6x$. Find the area of the shaded region.



Solution

Let A be the area of the shaded region below the x-axis and B the area of the shaded region above the x-axis.

Area of region $A = \int_0^3 y\,dx$

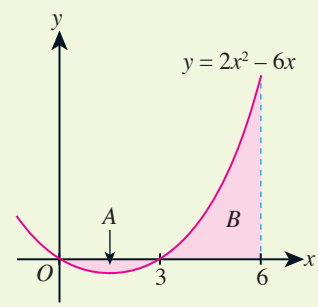
$= \int_0^3 (2x^2 - 6x)\,dx$

$= \left[\frac{2x^3}{3} - \frac{6x^2}{2}\right]_0^3$

$= \left[\frac{2(3)^3}{3} - 3(3)^2\right] - \left[\frac{2(0)^3}{3} - 3(0)^2\right]$

$= -9$

Hence, the area of region A is 9 units^2.



Area of region $B = \int_3^6 y\,dx$

$= \int_3^6 (2x^2 - 6x)\,dx$

$= \left[\frac{2x^3}{3} - \frac{6x^2}{2}\right]_3^6$

$= \left[\frac{2(6)^3}{3} - 3(6)^2\right] - \left[\frac{2(3)^3}{3} - 3(3)^2\right]$

$= 45$

Hence, the area of region B is 45 units^2.

Hence, the total area of the shaded regions $= 9 + 45$

$= 54$ units^2

Alternative Method

Area of the shaded regions

$= \left|\int_0^3 (2x^2 - 6x)\,dx\right| + \int_3^6 (2x^2 - 6x)\,dx$

$= |-9| + 45$

$= 9 + 45$

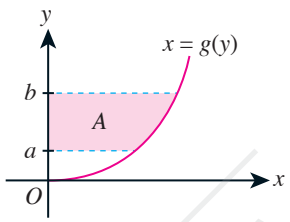
$= 54$ units^2

KEMENTERIAN PENDIDIKAN MALAYSIA

Area between the curve and the *y*-axis

The diagram on the right shows a region between the curve $x = g(y)$ and the y-axis bounded by the lines $y = a$ and $y = b$. The formula for the area of the region A is given by:

$$A = \int_a^b x \, dy$$



Discovery Activity 6

Group | 21st cl | STEM | CT

Aim: To determine the area of the region on the left and right of the y-axis

Steps:

bit.ly/3gTFYmj

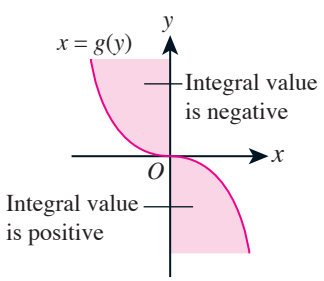
1. Scan the QR code on the right or visit the link below it.
2. Note the area bounded by the curve $x = y^{\frac{1}{3}}$ as shown on the graph.
3. Drag point a to $y = 0$ and point b to $y = 5$.
4. Take note of the location of the region formed and state whether the value of the area is positive or negative.
5. Repeat steps 3 and 4 and change point a to $y = -5$ and point b to $y = 0$.
6. Then, copy and complete the table below.

	The value of the integral	The location of the region
$\int_0^5 y^{\frac{1}{3}} \, dy$		
$\int_{-5}^0 y^{\frac{1}{3}} \, dy$		

7. Discuss with your group members regarding the signs of the definite integrals and their corresponding locations.
8. Present the group's findings to the class.

The result of Discovery Activity 6 shows that:

For a region bounded by the curve and the y-axis,
- If the region is to the left of y-axis, then the integral value is **negative**.
- If the region is to the right of y-axis, then the integral value is **positive**.
- The areas of both regions are positive.

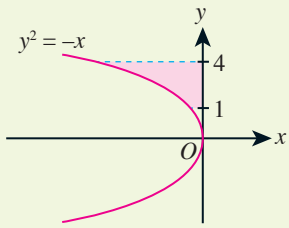


KEMENTERIAN PENDIDIKAN MALAYSIA

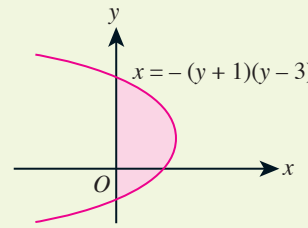
Example 16

Find the area for each of the following shaded regions.

(a)



(b)



Solution

(a) Given $y^2 = -x$.
So, $x = -y^2$.

$$\text{Area of the region} = \int_1^4 x\,dy$$
$$= \int_1^4 -y^2\,dy$$
$$= \left[-\frac{y^3}{3}\right]_1^4$$
$$= \left[-\frac{4^3}{3}\right] - \left[-\frac{1^3}{3}\right]$$
$$= -21$$

Thus, the area of the shaded region is 21 units^2.

Calculator Literate

To find the solution for Example 16(a) using a scientific calculator.

1. Press

$\int_{\square}^{\square}\square$ (–) ALPHA) x^2 ▶ 1 ▶ 4 =

2. The screen will display

$\int_1^4 -x^2\,dx$
−21

(b) Given $x = -(y + 1)(y - 3)$.
When $x = 0$,
$-(y + 1)(y - 3) = 0$
$y = -1$ or $y = 3$

Then, the shaded region is bounded by $y = -1$ and $y = 3$.

Because of that,

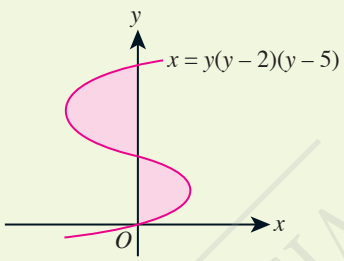
$$\text{Area of the region} = \int_{-1}^3 x\,dy$$
$$= \int_{-1}^3 -(y + 1)(y - 3)\,dy$$
$$= \int_{-1}^3 (-y^2 + 2y + 3)\,dy$$
$$= \left[-\frac{y^3}{3} + \frac{2y^2}{2} + 3y\right]_{-1}^3$$
$$= \left[-\frac{3^3}{3} + 3^2 + 3(3)\right] - \left[-\frac{(-1)^3}{3} + (-1)^2 + 3(-1)\right]$$
$$= 9 - \left(-\frac{5}{3}\right)$$
$$= \frac{32}{3}$$

Thus, the area of the shaded region is $\frac{32}{3}$ units^2.

KEMENTERIAN PENDIDIKAN MALAYSIA

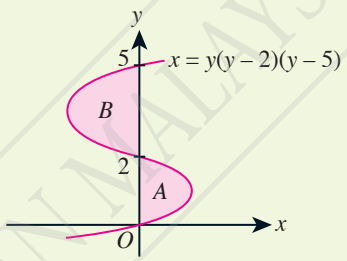
Example 17

The diagram shows a part of the curve $x = y(y - 2)(y - 5)$.
Find the area of the shaded regions.



Solution

Let A be the area of the region to the right of the y-axis and B the area of the region to the left of the y-axis.



Given $x = y(y - 2)(y - 5)$.
When $x = 0$,
$y(y - 2)(y - 5) = 0$
$y = 0, \quad y = 2 \quad \text{or} \quad y = 5$

Then, the region A is bounded by $y = 0$ and $y = 2$ and the region B is bounded by $y = 2$ and $y = 5$.

Because of that,

Area of region A

$$= \int_0^2 y(y - 2)(y - 5)\, dy$$

$$= \int_0^2 (y^3 - 7y^2 + 10y)\, dy$$

$$= \left[\frac{y^4}{4} - \frac{7y^3}{3} + \frac{10y^2}{2}\right]_0^2$$

$$= \left[\frac{2^4}{4} - \frac{7(2)^3}{3} + 5(2)^2\right] - \left[\frac{0^4}{4} - \frac{7(0)^3}{3} + 5(0)^2\right]$$

$$= \frac{16}{3} - 0$$

$$= \frac{16}{3}$$

Thus, the area of region A is $\frac{16}{3}$ units^2.

Area of region B

$$= \int_2^5 y(y - 2)(y - 5)\, dy$$

$$= \int_2^5 (y^3 - 7y^2 + 10y)\, dy$$

$$= \left[\frac{y^4}{4} - \frac{7y^3}{3} + \frac{10y^2}{2}\right]_2^5$$

$$= \left[\frac{5^4}{4} - \frac{7(5)^3}{3} + 5(5)^2\right] - \left[\frac{2^4}{4} - \frac{7(2)^3}{3} + 5(2)^2\right]$$

$$= -\frac{125}{12} - \frac{16}{3}$$

$$= -\frac{63}{4}$$

Thus, the area of region B is $\frac{63}{4}$ units^2.

Total area of the shaded region $= \frac{16}{3} + \frac{63}{4}$

$$= \frac{253}{12}$$

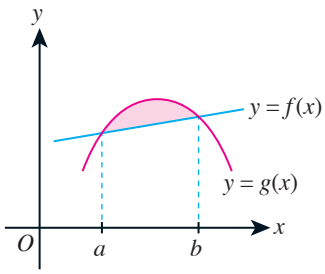
Hence, the area of the shaded regions is $\frac{253}{12}$ units^2.

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Area between the curve and a straight line

The shaded region shown in Diagram 3.1(a) is a region between the curve $y = g(x)$ and the straight line $y = f(x)$ from $x = a$ to $x = b$.

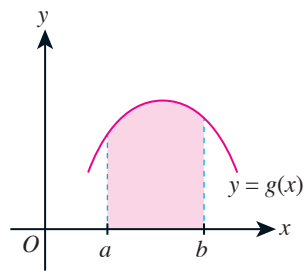
The area of the shaded region can be illustrated as follows:



The area of shaded region

Diagram 3.1(a)

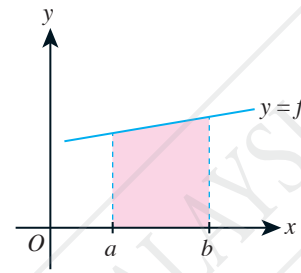
=



Area under the curve $y = g(x)$

Diagram 3.1(b)

−



Area under the straight line $y = f(x)$

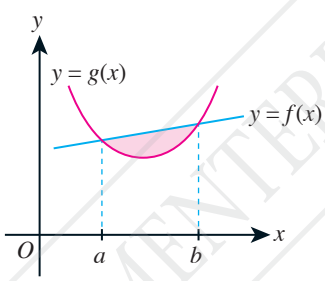
Diagram 3.1(c)

Thus,

$$\text{The area of shaded region} = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$
$$= \int_a^b [g(x) - f(x)]\,dx$$

The area of the shaded region in Diagram 3.2(a) shows the area between the straight line $y = f(x)$ and the curve $y = g(x)$ from $x = a$ to $x = b$.

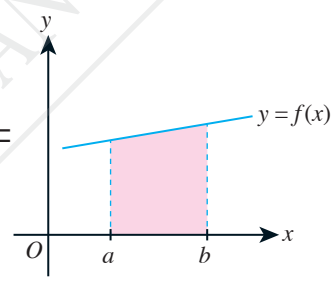
The area of the shaded region can be illustrated as follows:



The area of shaded region

Diagram 3.2(a)

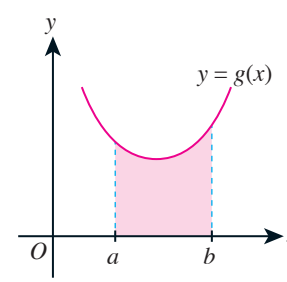
=



Area under the straight line $y = f(x)$

Diagram 3.2(b)

−



Area under the curve $y = g(x)$

Diagram 3.2(c)

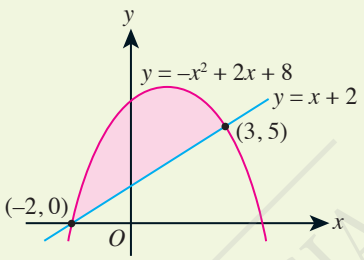
Thus,

$$\text{The area of shaded region} = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$
$$= \int_a^b [f(x) - g(x)]\,dx$$

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Example 18

In the diagram to the right, the curve $y = -x^2 + 2x + 8$ intersects the line $y = x + 2$ at points $(-2, 0)$ and $(3, 5)$. Find the area of the shaded region.



Solution

Area of the shaded region

$$= \int_{-2}^{3} (-x^2 + 2x + 8)\,dx - \int_{-2}^{3} (x + 2)\,dx$$

$$= \int_{-2}^{3} (-x^2 + 2x + 8 - x - 2)\,dx$$

$$= \int_{-2}^{3} (-x^2 + x + 6)\,dx$$

$$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x\right]_{-2}^{3}$$

$$= \left[-\frac{3^3}{3} + \frac{3^2}{2} + 6(3)\right] - \left[-\frac{(-2)^3}{3} + \frac{(-2)^2}{2} + 6(-2)\right]$$

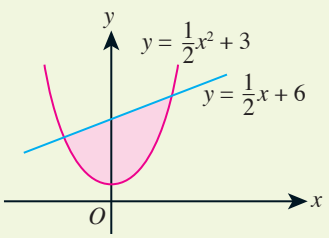
$$= \frac{125}{6} \text{ units}^2$$

DISCUSSION

Is there any other methods that can be used to solve Example 18? Discuss.

Example 19

The diagram on the right shows that the straight line $y = \frac{1}{2}x + 6$ intersects the curve $y = \frac{1}{2}x^2 + 3$. Calculate the shaded area bounded by the line and the curve.



Solution

$y = \frac{1}{2}x^2 + 3$ …❶

$y = \frac{1}{2}x + 6$ …❷

Substitute ❶ into ❷,

$$\frac{1}{2}x^2 + 3 = \frac{1}{2}x + 6$$

$$\frac{1}{2}x^2 - \frac{1}{2}x - 3 = 0$$

$$x^2 - x - 6 = 0$$

$$(x + 2)(x - 3) = 0$$

$$x = -2 \text{ or } x = 3$$

Area of the region

$$= \int_{-2}^{3} \left(\frac{1}{2}x + 6\right) dx - \int_{-2}^{3} \left(\frac{1}{2}x^2 + 3\right) dx$$

$$= \int_{-2}^{3} \left(\frac{1}{2}x + 6\right) - \left(\frac{1}{2}x^2 + 3\right) dx$$

$$= \int_{-2}^{3} \left(\frac{1}{2}x - \frac{1}{2}x^2 + 3\right) dx$$

$$= \left[\frac{x^2}{4} - \frac{x^3}{6} + 3x\right]_{-2}^{3}$$

$$= \left[\frac{3^2}{4} - \frac{3^4}{6} + 3(3)\right] - \left[\frac{(-2)^2}{4} - \frac{(-2)^3}{6} + 3(-2)\right]$$

$$= \frac{125}{12} \text{ units}^2$$

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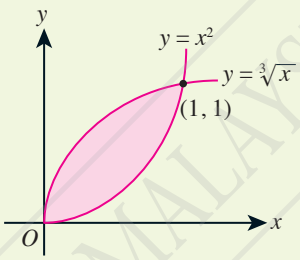
Area between two curves

Example 20

The curves $y = x^2$ and $y = \sqrt[3]{x}$ intersect at points (0, 0) and (1, 1). Find the area between the two curves.

Solution

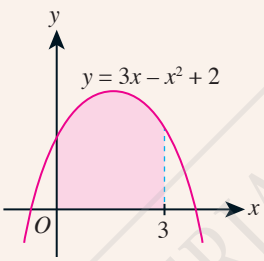
$$\text{Area of the region} = \int_0^1 \sqrt[3]{x}\, dx - \int_0^1 x^2\, dx$$
$$= \int_0^1 \left(x^{\frac{1}{3}} - x^2\right) dx$$
$$= \left[\frac{3x^{\frac{4}{3}}}{4} - \frac{x^3}{3}\right]_0^1$$
$$= \left[\frac{3(1)^{\frac{4}{3}}}{4} - \frac{1^3}{3}\right] - \left[\frac{3(0)^{\frac{4}{3}}}{4} - \frac{0^3}{3}\right]$$
$$= \frac{5}{12} \text{ unit}^2$$



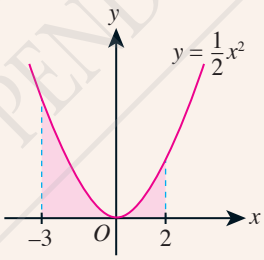
Self-Exercise 3.6

1. Find the area for each of the following shaded regions.

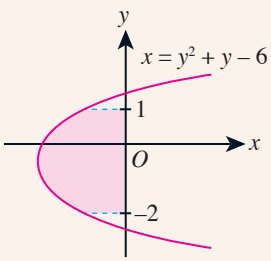
(a)



(b)

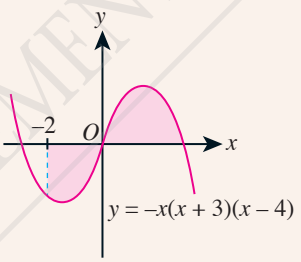


(c)

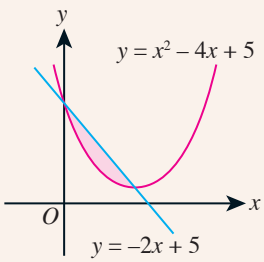


2. Find the area for each of the following shaded regions.

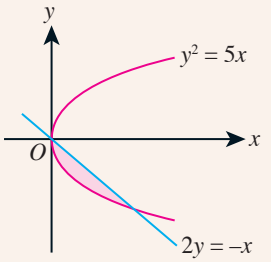
(a)



(b)



(c)



3. (a) If the curve $y = -x^3 - x^2$ intersects the curve $y = -x - x^2$ at points (–1, 0), (0, 0) and (1, –2), find the area between the two curves.

(b) Given that the curves $y = x^2 - 4x$ and $y = 2x - x^2$ intersect at two points, find the area of the region between the two curves.

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The relation between the limits of the sum of volumes of cylinders and the generated volume by revolving a region

Discovery Activity 7 Group

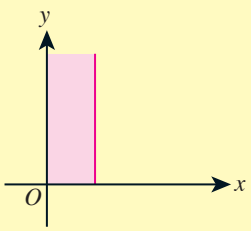
Aim: To determine the shape of a solid when a region is revolved 360° about an axis



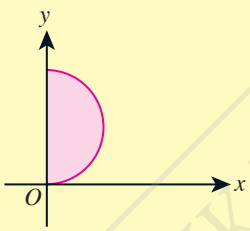
Steps:

1. Prepare three paper lanterns similar to the diagram shown.
2. Split the lanterns and take the largest part.
3. Take note of the three shaded regions below. Draw each onto three different lantern papers.

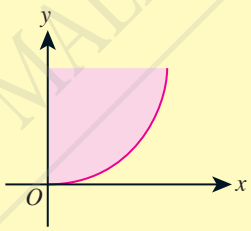
(a)



(b)



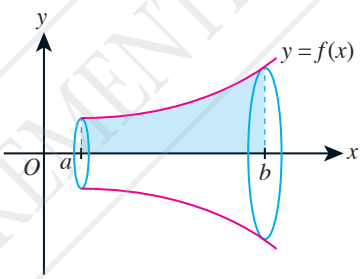
(c)

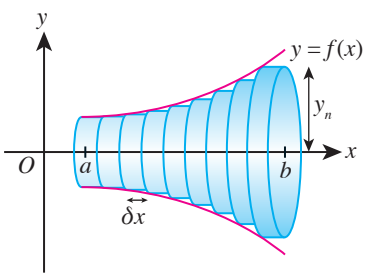


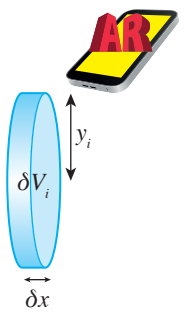
4. Cut each lantern paper according to the shaded region drawn on it.
5. Open them up and join the two ends.
6. Observe the three solids formed. What is the relation between the solids formed and the rotation through 360° of the paper pattern?

From Discovery Activity 7, a solid is generated when a region is revolved through 360° about an axis.

The generated volume of a solid when an area is rotated through 360° about the x-axis can be obtained by dividing the solid into n vertical cylinders with a thickness of δx as shown in the diagram below.







When the value of δx is small, the generated volume of the solid is the sum of all these cylinders.

Volume of each cylinder, δV_i = Area of the cross-section × Width of the cylinder

$$= \pi y_i^{\,2} \times \delta x$$
$$= \pi y_i^{\,2} \delta x$$

Volume of n cylinders $= \delta V_1 + \delta V_2 + \delta V_3 + \ldots + \delta V_n$

$$= \sum_{i=1}^{n} \delta V_i$$

$$= \sum_{i=1}^{n} \pi y_i^2 \, \delta x$$

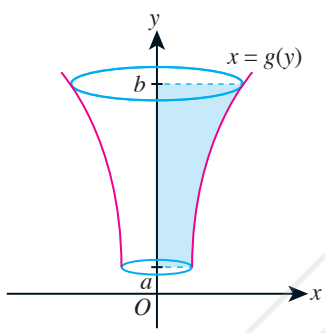
Information Corner

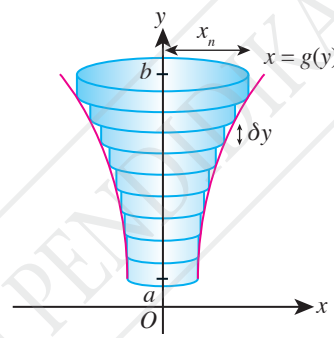
The value of generated volume is always positive.

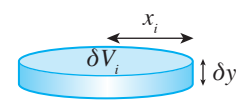
When the number of cylinders is sufficiently large, that is $n \to \infty$, then $\delta x \to 0$.
In general,

$$\text{The generated volume} = \lim_{\delta x \to 0} \sum_{i=1}^{n} \pi y_i^2 \, \delta x = \int_a^b \pi y^2 \, dx$$

The generated volume of a solid when a region is rotated through 360° about the y-axis can be determined in a similar manner as the generated volume of a solid when a region is rotated through 360° about the x-axis. The solid is divided into many n horizontal cylinders with thickness δy as shown in the diagram below.







When the value of δy is very small, the generated volume of the solid is the sum of all these cylinders.

Volume of each cylinder, δV_i = Area of the cross-section × Thickness of the cylinder

$$= \pi x_i^2 \times \delta y$$

$$= \pi x_i^2 \delta y$$

Volume of n cylinders $= \delta V_1 + \delta V_2 + \delta V_3 + \ldots + \delta V_n$

$$= \sum_{i=1}^{n} \delta V_i$$

$$= \sum_{i=1}^{n} \pi x_i^2 \, \delta y$$

When the number of cylinders is sufficiently large, that is $n \to \infty$, then $\delta y \to 0$.
In general,

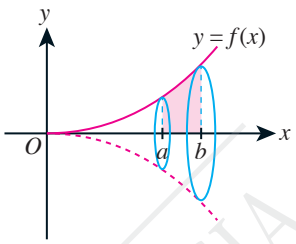
$$\text{The generated volume} = \lim_{\delta y \to 0} \sum_{i=1}^{n} \pi x_i^2 \, \delta y = \int_a^b \pi x^2 \, dy$$

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The generated volume of a region revolved at the *x*-axis or the *y*-axis

The generated volume V when a region bounded by the curve $y = f(x)$ enclosed by $x = a$ and $x = b$ is revolved through 360° about the x-axis is given by:

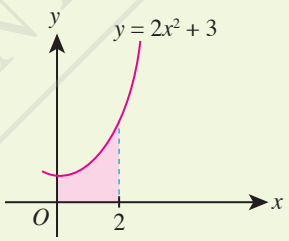
$$V = \int_a^b \pi y^2 \, dx$$



Example 21

Find the generated volume, in terms of π, when a region bounded by the curve $y = 2x^2 + 3$, the lines $x = 0$ and $x = 2$ is revolved through 360° about the x-axis.

Solution



$$\text{Generated volume} = \int_0^2 \pi y^2 \, dx$$

$$= \pi \int_0^2 (2x^2 + 3)^2 \, dx$$

$$= \pi \int_0^2 (4x^4 + 12x^2 + 9) \, dx$$

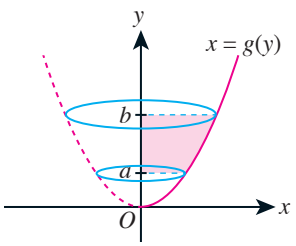
$$= \pi \left[\frac{4x^5}{5} + \frac{12x^3}{3} + 9x \right]_0^2$$

$$= \pi \left[\left(\frac{4(2)^5}{5} + 4(2)^3 + 9(2) \right) - \left(\frac{4(0)^5}{5} + 4(0)^3 + 9(0) \right) \right]$$

$$= 75\frac{3}{5}\pi \text{ units}^3$$

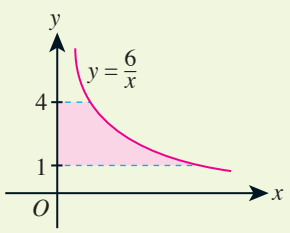
The generated volume V when a region bounded by the curve $x = g(y)$ enclosed by $y = a$ and $y = b$ is revolved through 360° about the y-axis is given by:

$$V = \int_a^b \pi x^2 \, dy$$



Example 22

Find the generated volume, in terms of π, when the shaded region in the diagram is rotated through 360° about the y-axis.



Solution

Given $y = \frac{6}{x}$

So, $x = \frac{6}{y}$

The generated volume $= \int_1^4 \pi x^2 \, dy$

$= \pi \int_1^4 \left(\frac{6}{y}\right)^2 dy$

$= \pi \int_1^4 \left(\frac{36}{y^2}\right) dy$

$= \pi \int_1^4 (36y^{-2}) \, dy$

$= \pi \left[\frac{36y^{-1}}{-1}\right]_1^4$

$= \pi \left[-\frac{36}{y}\right]_1^4$

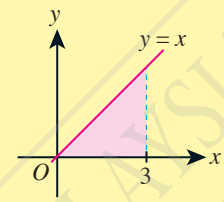
$= \pi \left[\left(-\frac{36}{4}\right) - \left(-\frac{36}{2}\right)\right]$

$= 27\pi \text{ units}^3$

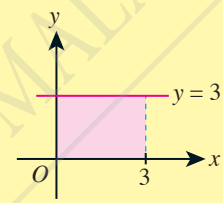
DISCUSSION

What geometrical shapes are formed when the following shaded areas in the diagram are revolved fully about the x-axis?

(a)



(b)

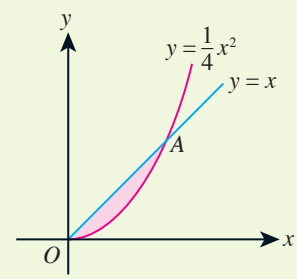


Example 23

In the diagram on the right, the curve $y = \frac{1}{4}x^2$ intersects the straight line $y = x$ at O and A. Find

(a) the coordinates of A,

(b) the generated volume, in terms of π, when the shaded region is revolved fully about the x-axis.



Solution

(a) $y = \frac{1}{4}x^2$ … ❶

$y = x$ … ❷

Substitute ❶ into ❷,

$\frac{1}{4}x^2 = x$

$x^2 = 4x$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x = 4$

Substitute $x = 4$ into ❷, we get $y = 4$.

Hence, the coordinates of A is $(4, 4)$.

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(b) Let V_1 be the volume generated by the straight line $y = x$ and V_2 be the volume generated by the curve $y = \frac{1}{4}x^2$ from $x = 0$ to $x = 4$.

$V_1 = \int_0^4 \pi(x)^2\,dx$

$V_1 = \pi\int_0^4 x^2\,dx$

$V_1 = \pi\left[\frac{x^3}{3}\right]_0^4$

$V_1 = \pi\left[\left(\frac{4^3}{3}\right) - \left(\frac{0^3}{3}\right)\right]$

$V_1 = \frac{64}{3}\pi$ units^3

$V_2 = \int_0^4 \pi\left(\frac{1}{4}x^2\right)^2\,dx$

$V_2 = \pi\int_0^4 \frac{1}{16}x^4\,dx$

$V_2 = \pi\left[\frac{x^5}{16(5)}\right]_0^4$

$V_2 = \pi\left[\left(\frac{4^5}{80}\right) - \left(\frac{0^5}{80}\right)\right]$

$V_2 = \frac{64}{5}\pi$ units^3

Thus, the generated volume $= V_1 - V_2$

$= \frac{64}{3}\pi - \frac{64}{5}\pi$

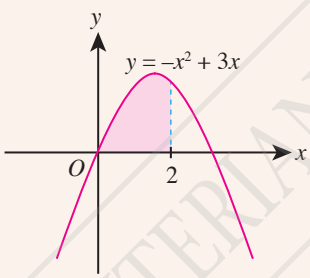
$= 8\frac{8}{15}\pi$ units^3

CHAPTER 3

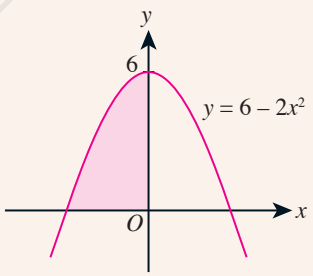
Self-Exercise 3.7

1. Find the generated volume, in terms of π, when the shaded region in each diagram is revolved through 360°:

(a) About the x-axis.



(b) About the y-axis.



2. Calculate the generated volume, in terms of π, when the enclosed region by the curve $y^2 = -4x$, $y = 0$ and $y = 2$ is revolved through 360° about the y-axis.

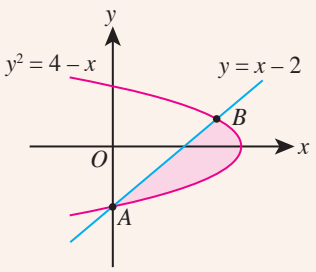
3. Find the generated volume, in terms of π, when the enclosed region by the straight line $y = 5 - x$, the curve $y = -x^2 + 4$, x-axis and y-axis is revolved fully about the x-axis.

4. In the diagram on the right, the curve $y^2 = 4 - x$ and the straight line $y = x - 2$ intersect at two points, A and B. Find

(a) the coordinates of A,

(b) the coordinates of B,

(c) the generated volume, in terms of π, when the enclosed shaded region by the curve $y^2 = 4 - x$ and the straight line $y = x - 2$ is rotated through 360° about the y-axis.



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Formative Exercise 3.3

Quiz bit.ly/30Twzq5

1. Find the value for each of the following.

(a) $\int_{-1}^{3} (2-x)^5\,dx$ (b) $\int_{-3}^{2} \frac{8x-6x^2+8}{2-x}\,dx$ (c) $\int_{-2}^{3} 2x^2(x^2-x)dx$

2. (a) Given $\int_{0}^{3} f(x)\,dx = 2$ and $\int_{2}^{5} g(x)\,dx = 7$, find the value of $\int_{3}^{0} \frac{1}{2} f(x)\,dx + \int_{2}^{5} 3g(x)\,dx$.

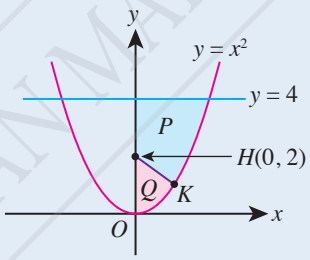
(b) If $\int_{1}^{7} k(x)\,dx = 10$, find the value of $\int_{1}^{3} [k(x)-3]\,dx + \int_{3}^{7} k(x)\,dx$.

3. Given the area under the curve $y = x^2 + hx - 5$ bounded by the lines $x = 1$ and $x = 4$ is $28\frac{1}{2}$ units^2, find the value of h.

4. The diagram on the right shows a curve $y = x^2$ and the straight line $y = 4$. A line with a gradient of -1 is drawn to pass through $H(0, 2)$ and it intersects the curve $y = x^2$ at K. Find

(a) the coordinates of K,

(b) the ratio of the area P to the area Q.



5. (a) Sketch the graph for the curve $y = 6x + x^2$.

(b) Find the equation of the tangents to the curve $y = 6x + x^2$ at the origin and at the point where $x = 2$.

(c) Given that the two tangents to the curve intersect at A, find the coordinates of A. Then, find the enclosed area by the tangents and the curve.

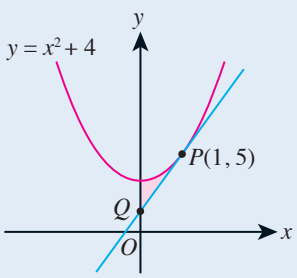
6. Find the generated volume, in terms of π, for the region bounded by the curve $y = x^2 + 2$, the lines $x = 1$ and $x = 2$ when it is rotated through 360° about the y-axis.

7. The diagram on the right shows a curve $y = x^2 + 4$ and the tangent to the curve at point $P(1, 5)$.

(a) Find the coordinates of Q.

(b) Calculate the area of shaded region.

(c) Calculate the generated volume, in terms of π, when the region bounded by the curve $y = x^2 + 4$, the y-axis and the line $y = 8$ is revolved fully about the y-axis.

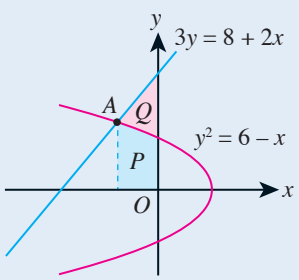


8. The diagram on the right shows a curve $y^2 = 6 - x$ and the straight line $3y = 8 + 2x$ that intersect at point A.

(a) Find the coordinates of the point A.

(b) Calculate the area of shaded region Q.

(c) Calculate the generated volume, in terms of π, when the shaded region P is rotated through 360° about the x-axis.



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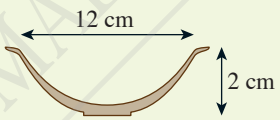
3.4 Applications of Integration

Integration is a branch of calculus and has many applications in our daily lives. Through integration, we can find the areas of regions formed by curves, determine the distance moved by an object from its velocity function and solve many other types of problems in various fields of economics, biology and statistics.

Solving problems involving integrations

Example 24 MATHEMATICAL APPLICATIONS

The diagram on the right shows the cross-section of a parabolic bowl whose function can be represented by $y = ax^2$. The diameter and the depth of the bowl are 12 cm and 2 cm respectively. Show that $a = \frac{1}{18}$. Subsequently, find the internal volume of the bowl, in terms of π.



Solution

1 . Understanding the problem

- The internal shape of the bowl is represented by $y = ax^2$.
- The diameter of the bowl = 12 cm.
- The depth of the bowl = 2 cm.
- Find the value of a for the equation $y = ax^2$.
- Find the generated volume, in terms of π, for the internal volume of the bowl.

2 . Planning the strategy

- Substitute the coordinates (6, 2) into the equation $y = ax^2$.
- Use the formula $\int_0^2 \pi x^2 \, dy$.

3 . Implementing the strategy

Given $y = ax^2$.

When $x = 6$ and $y = 2$,

$$2 = a(6)^2$$
$$2 = 36a$$
$$a = \frac{1}{18}$$

So, $y = \frac{1}{18}x^2$

$$x^2 = 18y$$

The internal volume of the bowl

$$= \int_0^2 \pi(18y) \, dy$$
$$= \pi\left[\frac{18y^2}{2}\right]_0^2$$
$$= \pi[9(2)^2 - 9(0)^2]$$
$$= 36\pi \text{ cm}^3$$

4 . Check and reflect

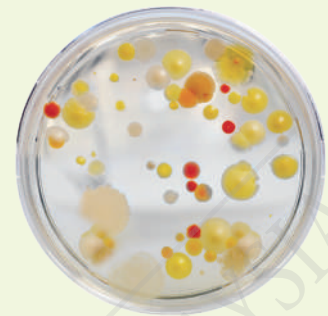
$$\int_0^2 \pi\left(\frac{y}{a}\right) dy = 36\pi$$
$$\pi\left[\frac{y^2}{2a}\right]_0^2 = 36\pi$$
$$\left[\frac{2^2}{2a} - \frac{0^2}{2a}\right] = \frac{36\pi}{\pi}$$
$$\frac{2}{a} = 36$$
$$a = \frac{1}{18}$$

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Example 25 MATHEMATICAL APPLICATIONS

In a research, it is found that the rate of growth of a colony of bacteria in a laboratory environment is represented by $\frac{dA}{dt} = 2t + 5$, where A is the area of the colony of bacteria, in cm^2, and t is the time, in seconds, for the bacteria to be cultured.

Given that the number of bacteria per 1 cm^2 is 1 000 000 cells and the colony of bacteria is only one cell thick, find the number of bacteria after 5 seconds.



Solution

1 . Understanding the problem

- Rate of increase of the bacteria colony in the laboratory is $\frac{dA}{dt} = 2t + 5$.
- Number of bacteria per 1 cm^2 = 1 000 000 cells.
- Find the area of the bacteria colony.
- Find the number of bacteria after 5 seconds.

2 . Planning the strategy

- Use the formula $\int_0^5 (2t + 5)\, dt$.
- Find the number of bacteria by multiplying the area of the bacteria colony with the number of cells per cm^2.

3 . Implementing the strategy

Area of the colony after 5 seconds

$$= \int_0^5 (2t + 5)\, dt$$

$$= \left[\frac{2t^2}{2} + 5t\right]_0^5$$

$$= \left[t^2 + 5t\right]_0^5$$

$$= [(5^2 + 5(5)) - (0^2 + 5(0))]$$

$$= 50 \text{ cm}^2$$

Number of bacteria $= 50 \times 1\,000\,000$

$= 50\,000\,000$

$= 5 \times 10^7$

Hence, the number of bacteria after 5 seconds is 5×10^7 cells.

4 . Check and reflect

Let u be the time taken to culture 5×10^7 cells bacteria.

$$\left[\int_0^u (2t + 5)\, dt\right] \times 1\,000\,000 = 5 \times 10^7$$

$$\left[\frac{2t^2}{2} + 5t\right]_0^u = \frac{5 \times 10^7}{1\,000\,000}$$

$$\left[t^2 + 5t\right]_0^u = \frac{5 \times 10^7}{1\,000\,000}$$

$$[(u^2 + 5u) - 0] = 50$$

$$u^2 + 5u = 50$$

$$u^2 + 5u - 50 = 0$$

By using factorisation, we get

$$(u + 10)(u - 5) = 0$$

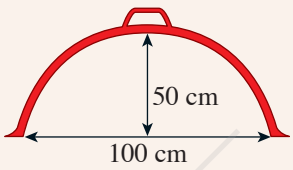
$$u = -10 \text{ or } u = 5$$

Since u must be positive, then $u = 5$ seconds.

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Self-Exercise 3.8

1. The diagram on the right shows the cross-section of a rattan food cover which is parabolic in shape and is presented by the equation $y = -kx^2$, where y is the height, in metres, and x is the radius of the food cover, in metres.



(a) Show that $k = \frac{1}{50}$.

(b) Find the internal volume of the food cover, in terms of π.

2. The yearly rate of depreciation of the price of a car is given by $S'(t) = \frac{A}{1\,000}(20 - t)$, where A is the original price of the car, in RM, and t is the number of years after being bought.

(a) Given that the original price of a car is RM48 000, find the price of the car after 7 years.

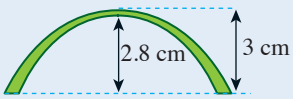
(b) If the original price of a car is RM88 500, find the percentage of depreciation of the car after 5 years.

Formative Exercise 3.4

Quiz bit.ly/3fV814h

1. A factory produces palm cooking oil. One of the cylindrical tanks containing the cooking oil is leaking. The height of the oil in the tank decreases at a rate of 5 cmmin^{-1} and the rate of change of the volume of the oil in the tank is given by $\frac{dV}{dh} = \frac{3}{5}t - 6$, where t is the time, in minutes. Find the volume, in cm^3, of the oil that has leaked out after 0.5 hour.

2. The diagram on the right shows the shape of the cross-section of a machine cover produced by a 3D printer. The cover is made from a kind of plastic. The internal and the external shapes of the cover are represented by the equations $y = -\frac{1}{16}x^2 + 2.8$ and $y = -\frac{1}{20}x^2 + 3$ respectively. Estimate the cost, in RM, of the plastic used to make the same 20 covers if the cost of 1 cm^3 of the plastic is 7 cents.



3. The rate of production of a certain machine by a factory is given by $\frac{dK}{dt} = 50\left[1 + \frac{300}{(t + 25)^2}\right]$, where K is the number of machines produced and t is the number of weeks needed to produce the machines. Find

(a) the number of machines produced after 5 years,

(b) the number of machines produced in the 6th year.

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REFLECTION CORNER

INTEGRATION

The reverse process of differentiation

Indefinite integral

- $\int ax^n \, dx = \frac{ax^{n+1}}{n+1} + c, \; n \neq -1$
- $\int [f(x) \pm g(x)] \, dx = \int f(x) \, dx \pm \int g(x) \, dx$
- $\int (ax + b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c, \; n \neq -1$

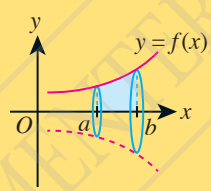
Definite integral

- $\int_a^b f(x) \, dx = \left[g(x) + c \right]_a^b = g(b) - g(a)$
- $\int_a^a f(x) \, dx = 0$
- $\int_a^b f(x) \, dx = -\int_b^a f(x) \, dx$
- $\int_a^b kf(x) \, dx = k\int_a^b f(x) \, dx$
- $\int_a^c f(x) \, dx = \int_a^b f(x) \, dx + \int_b^c f(x) \, dx$

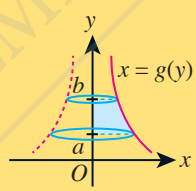
Equation of a curve

Given a gradient function $\frac{dy}{dx} = f'(x)$, then the equation of the curve for the function is $y = \int f'(x) \, dx$.

Generated volume

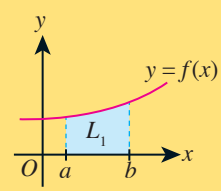


Generated volume $= \int_a^b \pi y^2 \, dx$

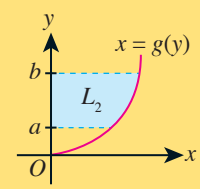


Generated volume $= \int_a^b \pi x^2 \, dy$

Area under a curve



Area of region L_1 $= \int_a^b y \, dx$



Area of region L_2 $= \int_a^b x \, dy$

Applications

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Journal Writing

Isaac Newton and Gottfried Wilhelm Leibniz were two mathematicians who were well known for their contributions to the field of calculus. However, both of them were involved in an intellectual dispute known as the Calculus Controversy.

Do a research on their contributions in the field of calculus and also the root cause of this controversy. Based on your findings, who was the first person who invented calculus? Present your results in an interesting graphic folio.

Summative Exercise

1. Find the indefinite integral for each of the following. PL 1

(a) $\int x(x-2)(x+3)\,dx$ (b) $\int \frac{2}{(2x-3)^3}\,dx$

2. It is given that $\int \frac{2}{(3x-2)^n}\,dx = a(3x-2)^{-2} + c$. PL 2

(a) Find the values of a and n.

(b) Using the value of n obtained in (a), find the value of $\int_1^3 \frac{8}{(3x-2)^n}\,dx$.

3. Given $y = \frac{3(2x+1)^2}{5x-1}$, show that $\frac{dy}{dx} = \frac{3(20x^2-8x-9)}{(5x-1)^2}$. Then, find the value of $\int_1^4 \frac{3(20x^2-8x-9)}{(5x-1)^2}\,dx$. PL 2

4. A curve has a gradient function $f'(x) = 2x^2 + 5x - r$, where r is a constant. If the curve passes through points $(1, 14)$ and $(-2, -16)$, find the value of r. PL 3

5. Given $\int_0^4 f(x)\,dx = 4$ and $\int_1^v g(x)\,dx = 3$, find PL 3

(a) the value of $\int_0^2 f(x)\,dx - \int_4^2 f(x)\,dx$,

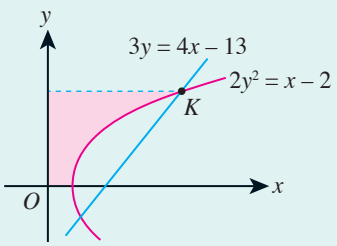
(b) the value of v if $\int_0^4 f(x)\,dx + \int_1^v [g(x) + x]\,dx = 19$.

6. It is given that $\frac{dV}{dt} = 10t + 3$, where V is the volume, in cm^3, of an object and t is time in seconds. When $t = 2$, the volume of the object is 24 cm^3. Find the volume, in cm^3, of the object when $t = 5$. PL 4

7. In the diagram on the right, the straight line $3y = 4x - 13$ intersects the curve $2y^2 = x - 2$ at point K. Find PL 2

(a) the coordinates of the point K,

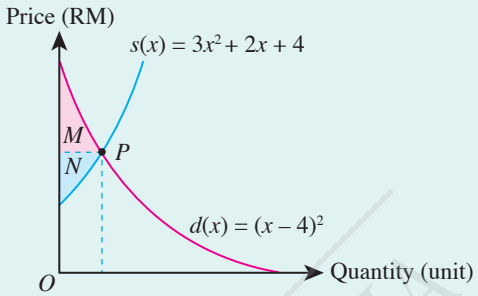
(b) the area of the shaded region.



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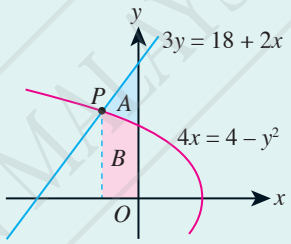
8. The diagram on the right shows a consumer demand curve, $d(x) = (x - 4)^2$ and a producer supply curve, $s(x) = 3x^2 + 2x + 4$. The region M represents the consumer surplus and the region N represents the producer surplus. The point P is known as an equilibrium point between the consumer demand and the producer supply. Find PL 3

(a) the equilibrium point P,
(b) the consumer surplus at the equilibrium point P,
(c) the producer surplus at the equilibrium point P.



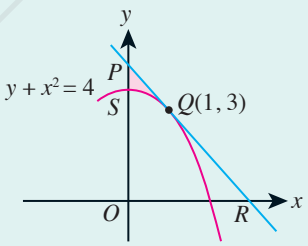
9. The diagram on the right shows a part of a curve $4x = 4 - y^2$ that intersects the straight line $3y = 18 + 2x$ at point P. PL 4

(a) Find the coordinates of the point P.
(b) Calculate the area of the shaded region A.
(c) Find the generated volume, in terms of π, when the shaded region B is rotated through 360° about the x-axis.



10. The diagram on the right shows a part of the curve $y + x^2 = 4$ and PR is a tangent to the curve at point $Q(1, 3)$. Find PL 4

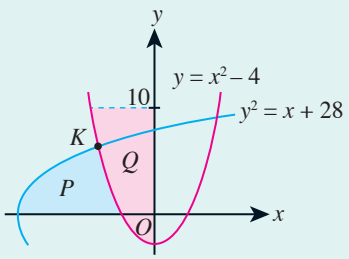
(a) the coordinates of the points P, R and S,
(b) the area of the shaded region,
(c) the generated volume, in terms of π, when the region bounded by the curve $y + x^2 = 4$, the y-axis and the straight line parallel to the x-axis and passes through the point Q is rotated through 360° about the y-axis.



11. Given a curve with the gradient function $f'(x) = px^2 + 6x$, where p is a constant. If $y = 24x - 30$ is the tangent equation to the curve at the point $(2, q)$, find the values of p and q. PL 4

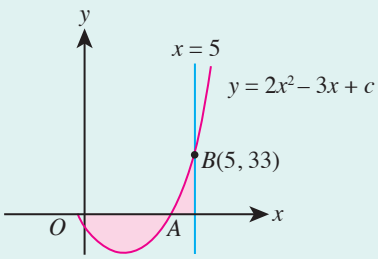
12. The diagram on the right shows the curve $y^2 = x + 28$ that intersects another curve $y = x^2 - 4$ at point $K(-3, 5)$. PL 4

(a) Calculate the area of the region P.
(b) Find the generated volume, in terms of π, when the region Q is rotated through 360° about the y-axis.



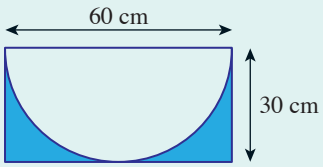
13. The diagram on the right shows a part of the curve $y = 2x^2 - 3x + c$ and the straight line $x = 5$. PL 4

(a) Find the value of c and the coordinates of point A.
(b) Calculate the area of the shaded region.
(c) Find the volume of revolution, in terms of π, when the region bounded by the curve $y = 2x^2 - 3x + c$ and the x-axis is rotated through 180° about the x-axis.



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14. The diagram on the right shows a cross-section of a container which has a parabolic inner surface and with a flat cover. The inner surface in the container can be represented by $y = ax^2$. Find the mass of rice, in kg, that can be stored in the container if the cover of the container can be tightly closed.
[Rice density = 1.182 g/cm^3] PL 4



15. Mr Razak plans to build a swimming pool at his residence. The swimming pool has a uniform depth of 1.2 m. PL 5

(a) The rate of flow of water to fill up the pool is given by $\frac{dV}{dt} = 3t^2 + 14t$, where V is the volume of water, in m^3, and t is the time, in hours. Mr Razak takes 5 hours to fill the water in the pool. Find the volume of the water inside the pool, in m^3.

(b) Mr Razak wants to paint the base of the pool with blue paint. The cost of painting is RM5 per m^2. If Mr Razak allocates RM1 000 for the cost of painting, can he paint the entire base of the pool? Give your reason.

CHAPTER 3

MATHEMATICAL EXPLORATION PBL

Scan the QR code on the right or visit the link below it to get the complete information on the project.

bit.ly/3gTMFFF

Introduction

Gold is a yellowish metal used as money for exchange, and has held a special value in human lives. The physical gold, which is shiny and does not oxidise even in water, makes the ornaments made from it appealing to many people.

Gold is also used in many other industries such as the manufacturing of computers, communication devices, space shuttles, jet engines, aircrafts and other products. The price of gold is constantly changing with time.

Reflection

Through the project, what did you learn? How can you apply your knowledge on integration in your daily life? Give your views by using an interesting graphic display.

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CHAPTER 4

PERMUTATION AND COMBINATION



Closed circuit television (CCTV)
IP:192.168.1.102



Mobile phone
IP:192.168.1.103





Printer
IP:192.168.1.1

What will be learnt?

- Permutation
- Combination

List of Learning Standards

bit.ly/3lLrNmT

Do you know that every computer or device that is connected to the Internet has its own Internet Protocol address (IP)? This Internet Protocol Address is created and managed by IANA (Internet Assigned Numbers Authority). In your opinion, how does a programmer select and arrange the Internet Protocol addresses for each device?



Computer
IP:192.168.1.100

Video about Internet Protocol (IP)

bit.ly/34MyV94

Info Corner

Al-Khalil Ahmad Al-Farahidi (718-791 C) was an Arabic mathematician and cryptographer who wrote 'Book of Cryptographic Messages'. In the book, permutation and combination were used for the first time to list all the possible Arabic words without vowels. His work in cryptography had also influenced Al-Kindi (801-873 C) who discovered the method of cryptoanalysis using the frequency analysis.

Cryptography is a study of linguistic that is related to secret codes, which can help a person to understand extinct languages.

For more info:

bit.ly/3epWiKh

Significance of the Chapter

- Normally, permutations and combinations are used in determining ATM pin numbers, security codes for mobile phones or computers or even in the matching of shirts and pants and others.
- It is extensively useful in the field of engineering, computer science, biomedical, social sciences and business.

Keywords

Product rule	*Petua pendaraban*
Permutations	*Pilih atur*
Factorial	*Faktorial*
Arrangement	*Susunan*
Order	*Tertib*
Combinations	*Gabungan*
Identical object	*Objek secaman*

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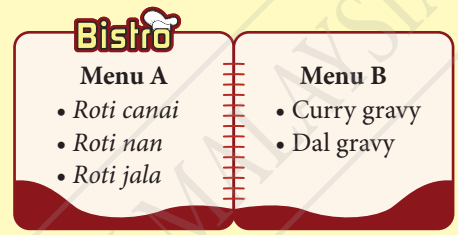
4.1 Permutation

Investigating and making generalisation on multiplication rule

Discovery Activity 1 Group 21st cl

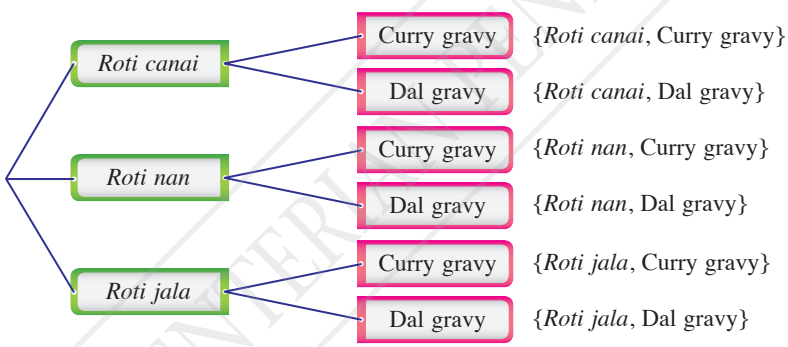
Aim: To investigate and make generalisation on the multiplication rule by using a tree diagram

Steps:



1. Your favourite shop offers breakfast sets. Based on the menu on the right, choose one type of bread to complement with one type of gravy.
2. By using a tree diagram, list out all the possible sets of your choice.
3. Then specify the number of ways that can be done.
4. Determine the number of choices if the shop also includes four types of drinks into the menu.
5. Discuss your findings among your group members and then appoint a representative from your group to present your group's findings to the class.

From the result of Discovery Activity 1, it is found that the number of choices can be illustrated by the tree diagram shown below.



There are six possible ways to choose a breakfast set. Besides listing out all the possible outcomes, an alternative way is to multiply together the possible outcomes of each event.

3 types of *roti* × 2 types of gravy = 6 ways to choose a breakfast set

If the shop includes another four types of drinks into the selection menu, then the number of ways will be:

3 types of *roti* × 2 types of gravy × 4 types of drinks = 24 ways to choose a breakfast set

The above method is known as the **multiplication rule**.

Information Corner

Hairi has 3 motorcycles and 2 cars. The number of ways for Hairi to use his vehicles to go to the store is as follows:

Motorcycle or Car

3 + 2 = 5 ways

A method used to determine the number of ways for events which are not sequential and are mutually exclusive is called the **addition rule**.

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In general,

Multiplication rule states that if an event can occur in m ways and a second event can occur in n ways, then both events can occur in $m \times n$ ways.

Example 1

(a) Determine the number of ways to toss a dice and a piece of coin simultaneously.
(b) Find the number of ways a person can guess a 4-digit code to access a cell phone if the digits can be repeated.

Solution

(a) The number of ways to toss a dice and a piece of coin simultaneously is $6 \times 2 = 12$.
(b) The number of ways a person can guess the 4-digit code to access a cell phone is $10 \times 10 \times 10 \times 10 = 10000$.

Information Corner

Multiplication rule can also be applied to more than two events.

DISCUSSION

Based on Example 1 (b), why is the solution given as $10 \times 10 \times 10 \times 10$? Explain.

CHAPTER 4

Self-Exercise 4.1

1. There are 3 choices of colours for a shirt while there are 5 choices of colours for a pair of pants. Determine the number of ways to match a shirt with a pair of pants.
2. How many ways are there to answer 15 true or false questions?
3. There are 4 roads joining Town A to Town B and 5 roads joining Town B to Town C. How many ways can a person travel to and fro through Town B if the person
   (a) uses the same roads? (b) does not use the same roads?

Determining the number of permutations

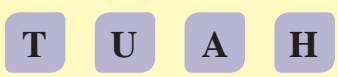
Determining the number of permutations for n different objects

Discovery Activity 2 Group 21st cl

Aim: To determine the number of permutations for n different objects arranged in a line

Steps:

1. Form a group of four or six members.
2. Each group will receive the word "TUAH" consisting of the letters T, U, A and H.



3. Each pupil will write an arrangement from the word TUAH on a piece of paper where duplication of letters are not allowed.
4. Then the paper is passed to the next person in the group to write another arrangement.
5. Repeat this process until there is no other possibilities available.
6. Then one of the members in the group will state the total number of possible arrangements.

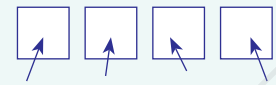
From Discovery Activity 2, there are two methods to find the number of ways to arrange the letters from the word TUAH where the letters are not repeated.

Method 1

List all the possible arrangements. In this activity, there are 24 ways you can arrange the letters without repetition.

Method 2

Fill in the empty boxes below.



4 choices 3 choices 2 choices 1 choice

From the second method:

- For the first box, there are four ways to fill in the box either with T, U, A or H.
- For the second box, there are three ways, the third box has two ways and the fourth box has only one way.
- By using multiplication rule, the number of possible ways is $4 \times 3 \times 2 \times 1 = 24$.

The number of ways to arrange these letters is called a **permutation**. $4 \times 3 \times 2 \times 1$ is also known as factorial and can be written as 4!. In general,

The number of permutations of n objects is given by $n!$, where $n! = {}^{n}P_{n} = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$.

DISCUSSION

Given $1! = 1$.
Explain why $0! = 1$.

Calculator Literate

To determine the permutation of 4 different objects by using a calculator.

1. Press
   4 SHIFT x^{-1} =
2. The screen will display
   4!
   24

Example 2

Without using a calculator, find the value of each of the following.

(a) $\frac{11!}{9!}$ (b) $\frac{6!}{4!2!}$

DISCUSSION

Simplify the following:
(a) $\frac{n!}{(n-2)!}$ (b) $\frac{(n-1)!}{n!}$

Solution

(a) $\frac{11!}{9!} = \frac{11 \times 10 \times \cancel{9!}}{\cancel{9!}}$
$= 11 \times 10$
$= 110$

(b) $\frac{6!}{4!2!} = \frac{6 \times 5 \times \cancel{4!}}{\cancel{4!} \times 2 \times 1}$
$= \frac{6 \times 5}{2 \times 1}$
$= 15$

Example 3

Find the number of ways to arrange all the letters from the word BIJAK when repetition of letters is not allowed.

Solution

Given the number of letters, $n = 5$.
Thus, the number of ways to arrange all the letters is $5! = 120$.

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Discovery Activity 3 Group 21st cl

Aim: To determine the number of permutations of n different objects in a line and in a circle

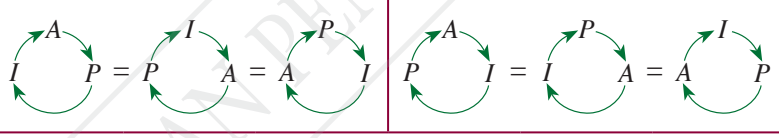
Steps:

1. Form groups consisting of six members.
2. Each group will be given a three-letter word as shown.

A	P	I

3. Each group is required to list out all the possible arrangements if the letters are arranged
   (a) in a line (b) in a circle
4. Take note of the linear and circular arrangements. Are the number of arrangements the same or different? What is the relation between permuting objects linearly and in a circle? Explain.
5. Discuss your group's findings and get your group's representative to present to the class.

From Discovery Activity 3, it is found that when the letters from the word API is arranged in a line, the number of possible ways is 3! = 6. If they are arranged in a circle, it is found that 3 of the linear permutations is the same as 1 permutation when arranged in a circle.

Types of arrangement	Arrangement						Number of arrangements
Linear	API	IAP	PIA	AIP	PAI	IPA	6
Circular							2

Hence the number of arrangements for the letters from the word API in a circle is $\frac{3!}{3} = 2$.

In general, the permutation of n objects in a circle is given by:

$$\frac{n!}{n} = \frac{n(n-1)!}{n} = (n-1)!$$

Example 4

Determine the number of ways to arrange six pupils to sit at a round table.

Solution

Given the number of pupils, $n = 6$. Thus, the number of ways to arrange the six pupils is $(6 - 1)! = 120$.

QR Access

Video to show how to arrange six pupils to sit at a round table

bit.ly/2QiGcIg

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Example 5

Find the number of ways to assemble 12 beads of different colours to form a toy necklace.

Solution

Given the number of beads, $n = 12$ and the beads are arranged in a circle. It is found that the clockwise and anticlockwise arrangements look the same.
So, the number of ways to arrange 12 beads is
$\frac{(12-1)!}{2} = \frac{11!}{2} = 19\,958\,400.$

Information Corner

The arrangements of objects in a circular bracelet or necklace do not involve clockwise or anticlockwise directions because both are the same. The number of arrangements is like arranging n objects in a circle and divide by 2, that is, $\frac{(n-1)!}{2}$.

Self-Exercise 4.2

1. Without using a calculator, find the value of each of the following.

(a) $\frac{8!}{5!}$ (b) $\frac{8!-6!}{6!}$ (c) $\frac{4!}{2!2!}$ (d) $\frac{7!5!}{4!3!}$

2. Find the number of ways to arrange all the letters from the following words without repetition.

(a) SURD (b) LOKUS (c) VEKTOR (d) PERMUTASI

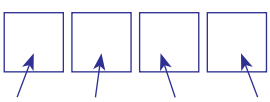
3. What is the number of ways to arrange seven customers to sit at a round table in a restaurant?

4. Determine the number of ways to arrange eight gemstones with different colours to form a chain.

Determining the number of permutations of n different objects, taking r objects each time

You have learnt how to calculate the number of ways to arrange four letters of the word TUAH by filling in the empty boxes and so obtaining $4 \times 3 \times 2 \times 1 = 24$ number of ways.

Consider the word BERTUAH. Suppose we want to arrange only four of these letters from the word into the boxes on the right.



7 choices 6 choices 5 choices 4 choices

In the first box, there are 7 ways to fill the letters. Then, the second box has 6 ways, the third box has 5 ways and the fourth box has 4 ways.

By using multiplication rule, the number of possible ways is $7 \times 6 \times 5 \times 4 = 840$.

The number of permutations for 7 different objects, taking 4 objects each time, can be represented by the notation ${}^7P_4$.

Note that $7 \times 6 \times 5 \times 4$ can also be written as:
$\frac{7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{7!}{3!} = \frac{7!}{(7-4)!}$
So, ${}^7P_4 = \frac{7!}{(7-4)!} = 840.$

In general,

The number of permutations of n objects taking r each time is given by ${}^nP_r = \frac{n!}{(n-r)!}$ where $r \leqslant n$.

Example 6

Without using a calculator, find the value of ${}^6P_4$.

Solution

$${}^6P_4 = \frac{6!}{(6-4)!} = \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times \cancel{2} \times \cancel{1}}{\cancel{2} \times \cancel{1}} = 360$$

DISCUSSION

Determine the following values of n.

(a) ${}^nP_2 = 20$

(b) ${}^{n+2}P_3 = 30n$

(c) ${}^{n+1}P_4 = 10\,{}^nP_2$

Example 7

Eight committee members from a society are nominated to contest for the posts of President, Vice President and Secretary. How many ways can this three posts be filled?

Solution

Three out of the eight committee members will fill up the three posts.

Hence, we have, ${}^8P_3 = \frac{8!}{(8-3)!} = 336$.

Calculator Literate

Using a scientific calculator to find the answer for Example 7.

1. Press

   8 SHIFT × 3 =

2. The screen will display

   8P3
   336

Consider the following situation.

Let's say four letters from the word BERTUAH need to be arranged in a circle, what is the number of ways to do this?

If the letters from the word BERTUAH is arranged in a line, the number of ways is ${}^7P_4 = 840$. However, if they are arranged in a circle, four of the arrangements are identical. Therefore, the number of ways to arrange 4 out of 7 letters in a circle is $\frac{{}^7P_4}{4} = \frac{840}{4} = 210$.

In general,

The number of permutations for n different objects taking r objects each time and arranged in a circle is given by $\frac{{}^nP_r}{r}$.

Excellent Tip

A permutation of an object in a circle where clockwise and anticlockwise arrangements are the same, then the number of ways is as follows.

$$\frac{{}^nP_r}{2r}$$

Example 8

Nadia bought 12 beads of different colours from Handicraft Market in Kota Kinabalu and she intends to make a bracelet. Nadia realises that the bracelet requires only 8 beads. How many ways are there to make the bracelet?

Solution

Given the number of beads is 12 and 8 beads are to be arranged to form a bracelet. It is found that clockwise and anticlockwise arrangements are identical.

Hence, the number of permutations is $\frac{{}^{12}P_8}{2(8)} = \frac{{}^{12}P_8}{16} = 1\ 247\ 400$.

Self-Exercise 4.3

1. Without using a calculator, find the value of each of the following.
(a) ${}^5P_3$ (b) ${}^8P_7$ (c) ${}^9P_5$ (d) ${}^7P_7$

2. In a bicycle race, 9 participants are competing for the first place, the first runner-up and the third place. Determine the number of permutations for the first three places.

3. A stadium has 5 gates. Find the number of ways 3 people can enter the stadium, each using different gates.

4. Find the number of ways to form four-digit numbers from the digits 2, 3, 4, 5, 6, 7, 8 and 9 if the digits cannot be repeated.

5. An employee at a restaurant needs to arrange 10 plates on a round table but the table can only accommodate 6 plates. Find the number of ways to arrange the plates.

Determining the number of permutations for *n* objects involving identical objects

Discovery Activity 4 Group 21st cl

Aim: To determine the number of permutations for *n* objects involving identical objects

Steps:

1. Each group is given one word consisting of three letters as follows.

A	P	A

2. Label the two letters A as A_1 and A_2 respectively, then construct a tree diagram.

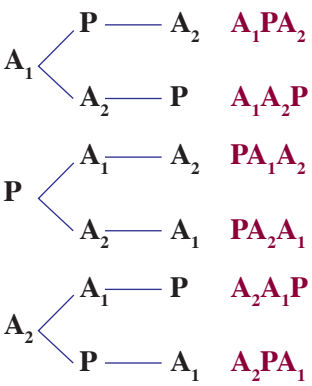
3. Based on the tree diagram, list all the possible arrangements of the letters. How many arrangements are there?

4. When A_1 and A_2 are the same, what is the number of arrangements? What method can be used to find the number of arrangements for words involving identical letters such as the letter A in the word APA?

5. Appoint a representative and present the findings of your group to the class.

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From Discovery Activity 4, the following results are obtained.



Number of arrangement = $\boxed{3}\boxed{2}\boxed{1}$
$= 3 \times 2 \times 1$
$= 6$
$= {}^{3}P_{3}$
$= 3!$

Technology Click

Explore the following GeoGebra to see the graphical representation of permutations of identical objects.

ggbm.at/arvybfjg

When $A_1 = A_2 = A$, where two arrangements are considered as one arrangement, 3 arrangements are obtained, namely APA, AAP and PAA. The method to obtain 3 ways of arrangement is by dividing the total number of arrangements of letters in A_1PA_2 by the number of arrangements of the 2 identical letters, A that is, $\frac{3!}{2!} = 3$.

In general,

The number of permutations for n objects involving identical objects is given by $P = \frac{n!}{a!b!c!\ldots}$, where a, b and c, … are the number of identical objects for each type.

Example 9

Calculate the number of ways to arrange the letters from the word SIMBIOSIS.

Solution

Given $n = 9$. The identical objects for letters S and I are the same, which is 3. Hence the number of ways to arrange the letters from the word SIMBIOSIS is $\frac{9!}{3!3!} = 10\,080$.

DISCUSSION

Suppose the letters from the word SIMBIOSIS is to be arranged starting with the letter S. How do you determine the number of ways to arrange those letters?

Self-Exercise 4.4

1. Determine the number of ways to arrange all the letters differently for each of the following words.
(a) CORONA
(b) MALARIA
(c) HEPATITIS
(d) SKISTOSOMIASIS

2. There are 5 blue pens and 3 red pens in a container. Find the number of ways to arrange all the pens in one line.

3. There are 4 white flags and 6 yellow flags inside a box. Find the number of ways to attach the flags in a line on a vertical pole.

4. Find the number of odd numbers that can be formed from all the numbers 3, 4, 6 and 8 with all the numbers other than 3 appearing exactly twice.

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Solving problems involving permutations with certain conditions

Consider seven objects in the diagram below.



Suppose all the above objects are to be arranged according to a certain condition. Then, the following conditions should be followed.

1

If each circle and triangle must be arranged alternatively,



- There are 4! = 24 ways to arrange four circles.
- There are 3! = 6 ways to arrange three triangles.
- By using multiplication rule, the number of possible ways to arrange the circles and triangles alternatively is 4! × 3! = 144.

2

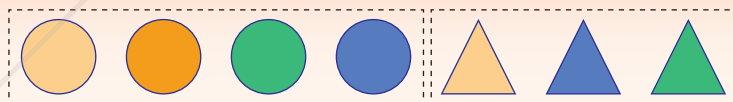
If all the circles are always together,



- There are 4! = 24 ways to arrange a group of circles and three triangles.
- There are 4! = 24 ways to arrange among the group of circles.
- By using multiplication rule, the number of possible ways is 4! × 4! = 576.

3

If circles and triangles have to be arranged in their respective groups,



- There are 4! × 3! = 144 ways to arrange all the circles together in front of the line to be followed by the three triangles.



- Every object can also be arranged such that all triangles are together in front of the line followed by the four circles, which is 3! × 4! = 144.
- Hence the total possible number of ways is 144 + 144 = 288.

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Example 10

Find the number of ways to form 4-digit odd numbers from the digits 1, 3, 4, 5, 6, 8 and 9 without repeating any of the digits.

Solution

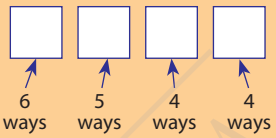
To form an odd number, it must end with an odd digit.
There are four ways to fill the last digit, that is, with either 1, 3, 5 or 9.



After one of the odd digits has been used, there are six more numbers which can be used to fill up the front three digits, thus ${}^{6}P_{3} \times {}^{4}P_{1} = 480$.
Hence there are 480 ways to form 4-digit odd numbers that fulfil the condition.

Alternative Method

Consider the number of ways to fill up each box below.



Total number of ways to fill up all the boxes is $6 \times 5 \times 4 \times 4 = 480$.
Hence there are 480 ways to form 4-digit odd numbers that fulfil the condition.

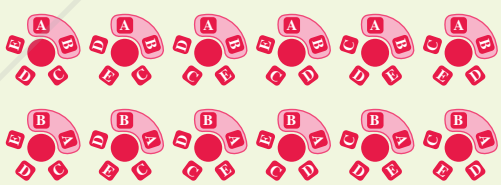
CHAPTER 4

Example 11

Find the number of ways to arrange 5 employees, A, B, C, D and E from a company at a round table if A and B must be seated together.

Solution

When A and B are seated together, they are regarded as one unit. Then the number of ways to arrange one unit of A and B and three others is $(4 - 1)! = 6$ ways.
A and B can interchange among themselves in 2! ways. Hence, the total permutations are $6 \times 2 = 12$ ways.



Example 12

Find the number of possible ways to arrange all the letters in the word SUASANA if the vowels are always together.

Solution

Given that the number of letters, $n = 7$ and the number of identical letters, S and A are 2 and 3 respectively. For the condition that the vowels are always together, group the vowels to form one unit.

(AAAU) S S N

So, the number of arrangements together with the other 3 letters is $\frac{4!}{2!}$ way.

In the group of vowels, there are 4 letters that can be arranged in $\frac{4!}{3!}$ ways.

Thus, the number of arrangements when the vowels are always together is $\frac{4!}{2!} \times \frac{4!}{3!} = 48$.

Example 13 MATHEMATICAL APPLICATIONS

Find the number of ways to form 4-digit numbers from the digits 2, 3, 5 and 7 if the numbers must be odd and less than 5 000.

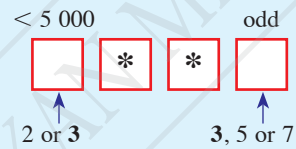
Solution

1 . Understanding the problem

Two conditions to form the 4-digit number from the digits 2, 3, 5 and 7 are it must be odd and less than 5 000.

2 . Planning the strategy

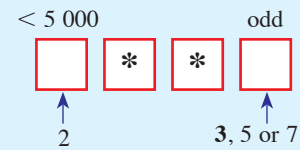
- To form the 4-digit numbers, prepare four empty boxes.
- For the number to be odd, the last digit must be odd.
- For the number to be less than 5 000, the first box consists of a digit that is less than 5.



3 . Implementing the strategy

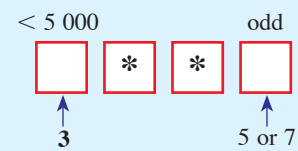
Case 1: If 3 is used for the last box.

- Then the first box has only one choice and the last box has 3 choices.
- The middle two boxes will have 2! ways.
  Thus there are $1 \times 2 \times 1 \times 3 = 6$ ways.



Case 2: If 3 is used in the first box.

- Then the first box has only one choice and the last box has 2 choices.
- The middle two boxes will have 2! ways.
  Thus there are $1 \times 2 \times 1 \times 2 = 4$ ways.



Thus the number of permutations = 6 + 4 = 10

Hence the total number of ways to form 4-digit numbers from the digits 2, 3, 5 and 7 where the numbers must be odd and less than 5 000 is 10.

4 . Check and reflect

Case 1: $1 \times {}^{2}P_{1} \times 3 = 6$

Case 2: $1 \times {}^{2}P_{1} \times 2 = 4$

Hence the number of permutations is 6 + 4 = 10.

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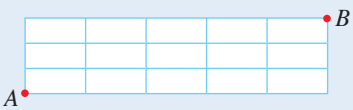
Self-Exercise 4.5

1. Find the number of ways in which the letters from the word TULAR can be arranged if
   (a) the vowels and the consonants are arranged alternatively,
   (b) each arrangement begins and ends with a vowel,
   (c) the consonants and the vowels are in their respective groups.
2. Find the number of ways for 4-digit numbers greater than 2 000 to be formed by using the digits 0, 2, 4, 5, 6 dan 7 without repetition.
3. Find all the possible arrangements of using all the letters in the word TRIGONOMETRI if G is the first letter and E is the last letter.
4. A family consisting of a father, a mother and 4 children are seated at a round table. Find the number of different ways they can be seated if
   (a) there are no conditions,
   (b) the father and the mother are seated together.

Formative Exercise 4.1

Quiz bit.ly/2Frhg00

1. A set of questions contains 5 true or false questions and 5 multiple choice questions each with four choices. What is the number of ways to answer this set of questions?
2. Find the number of ways to create a 3-digit password for a lock if
   (a) repetition of digits is allowed,
   (b) repetition of digits is not allowed.
3. How many numbers are there between 5 000 and 6 000 that can be formed from the digits 2, 4, 5, 7 and 8 without repetition of digits? How many of these are even numbers?
4. A couple and their eight children are going to watch a movie in cinema. They booked a row of seats. Find the number of ways the family can be seated if the couple
   (a) sit side by side,
   (b) sit at both ends of the row,
   (c) sit separately.
5. Find the number of ways to arrange each word BAKU and BAKA if no repetition is allowed. Are the number of ways the same? Explain.
6. Determine the number of routes for an object to move from point *A* to point *B* if the object can only move up or to the right.



7. A group of 7 children are competing for six chairs that are arranged in a circle during a musical chair game. The children have to move in an anticlockwise direction around the chairs. Determine the number of arrangements for this game.

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4.2 Combination

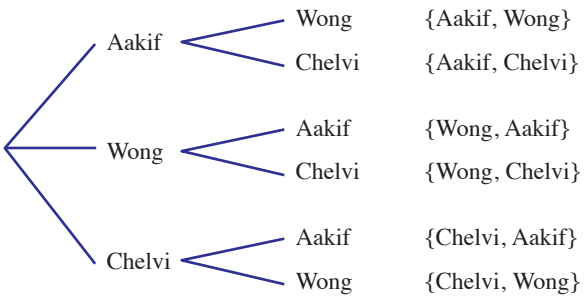
Comparing permutation and combination

In permutations, you have learnt that the position of each object in each set is important. For example, the arrangements AB and BA are two different arrangements.

Consider the problem below.

> Let's say you have three friends, Aakif, Wong and Chelvi. You need to choose two out of your three friends to join you in a kayaking activity.
> How many ways can you make this selection? Are your friends' positions important in this election?

By using a tree diagram, we can list out all the possible choices.



However, is the decision to choose 'Aakif and Wong' different from choosing 'Wong and Aakif'? In the above situation, is the position of an object important in making the choice?

Based on the diagram on the right, there are only 3 ways to choose since the positions of the objects are not important. Hence, the possible choices are {Aakif, Wong}, {Aakif, Chelvi} or {Wong, Chelvi}.

In general,

> When choosing an object from a set where positions or arrangements are not important, the selection is called **combination**.



State whether the following situations involve permutation or combination. Explain.

> A television station company offers to its customers a selection of 7 channels from the 18 available channels.

Determining the number of combinations of *r* objects chosen from *n* different objects at a time

Let's explore how to find the number of combinations of *r* objects chosen from *n* different objects at a time.

Discovery Activity 5

Pair 21st cl

Aim: To determine the number of combinations of *r* objects chosen from *n* different objects at a time

bit.ly/33rVzow

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Observe the four objects which are pictures of animals in the worksheet provided. Those objects will be hung to decorate your classroom.
3. In pairs, list the number of ways to hang each object based on the following conditions.
   (a) The arrangements must take into account the positions of the objects.
   (b) The arrangements do not take into account the positions of the objects.
4. Identify the number of ways if you and your partner are chosen to hang up
   (a) one object only,
   (b) two objects only,
   (c) three objects only.
5. Compare the results obtained in steps 3(a) and 3(b). Then, circle the list that has the same objects but with different arrangements.
6. What differences do you see between the two methods of hanging the pictures in terms of arrangements and the number of ways to do it?

CHAPTER 4

From the result of Discovery Activity 5, it shows that three out of four objects have been selected to be hung in the class.

- If positions are taken into account, then ${}^4P_3 = \frac{4!}{(4-3)!} = 24$.
- If the positions are ignored, there are $3! = 6$ groups that have the same objects. Therefore, the number of ways to select the objects to hang without taking the positions into account is

$24 \div 6 = 4$ or $\frac{4!}{3!(4-3)!} = 4$ or $\frac{{}^4P_3}{3!} = 4$.

In general, the number of combinations of *r* objects selected from *n* different objects is given by:

$$ {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{r!(n-r)!} $$

Information Corner

Combination can be written as ${}^nC_r$ or $\binom{n}{r}$.
${}^nC_r$ is also known as binomial coefficient.

Flash Quiz

Prove that ${}^nC_0 = 1$ and ${}^nC_1 = n$, where n is a positive integer.

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Example 14

The martial arts team of SMK Sari Baru consists of 8 pupils. 2 pupils will be selected to represent the team in a martial arts show. Determine the number of ways to choose the 2 pupils.

Solution

2 representatives are to be selected from the martial arts team consisting of 8 members.

So, the number of ways $= {}^{8}C_{2} = \frac{8!}{2!(8-2)!} = \frac{8!}{2!6!} = \frac{\cancel{8} \times 7 \times 6!}{\cancel{2} \times 1 \times 6!} = 28.$

Example 15

3 committee members are to be selected from 10 candidates in a club. Find the number of ways to select these committee members.

Solution

3 committee members need to be selected out of the 10 candidates.

So, the number of ways $= {}^{10}C_{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = 120.$

DISCUSSION

Compare Example 15 with Example 7. State the difference between the two questions which results in Example 7 to use permutation while Example 15 to use combination.

Example 16

Find the number of triangles that can be formed from the vertices of a hexagon.

Solution

Hexagon has six vertices. To form a triangle, any three vertices are required.

So, the number of ways $= {}^{6}C_{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!3!} = 20.$

Self-Exercise 4.7

1. There are 12 players in the school handball team. Determine the number of ways a coach can choose 5 players
(a) as striker 1, striker 2, striker 3, defender 1 and defender 2,
(b) to play in a district level competition.

2. Class 5 Al-Biruni has 25 pupils. Three representatives from the class are selected to attend a motivational camp. Find the number of ways to select the representatives.

3. What is the number of ways to select four letters from the letters P, Q, R, S, T and U?

4. *ABCDEFGH* are the vertices of a regular octagon. Find the number of diagonals that can be formed from the octagon.

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Solving problems involving combinations with certain conditions

Consider the situation below.

A class monitor wants to divide your 10 friends into three groups of two people, three people and five people. Find the number of ways the groupings can be done.

To solve a problem which involve combinations with certain conditions (conditions should be dealt with first)

Group 1	Group 2	Group 3
Select two out of 10 people.	• Two people have been taken by Group 1. • There are eight people left. • Select three out of eight people.	• Five people have been taken by Group 1 and Group 2. • There are only five people left. • Select five out of five people.
${}^{10}C_2 = \frac{10!}{2!(10-2)!}$ $= 45$ So, the number of ways is 45.	${}^{8}C_3 = \frac{8!}{3!(8-3)!}$ $= 56$ So, the number of ways is 56.	${}^{5}C_5 = \frac{5!}{5!(5-5)!}$ $= 1$ So, the number of ways is 1.

By using multiplication rule, the total number of ways is $45 \times 56 \times 1 = 2\,520$.

DISCUSSION

If you choose either five people first or three people first, will you get a different answer? Compare your answer with your friend's.

Technology Click

Graphic representation to find the number of combination

ggbm.at/hzzb4nwt

Example 17

A football team is made up of 17 local players and three foreign players. A coach wants to select 11 key players to compete in a match by including two foreign players. Find the number of ways to select these 11 players.

Solution

Number of ways to select two out of three foreign players, ${}^{3}C_2$.
Number of ways to select nine out of 17 local players, ${}^{17}C_9$.

Therefore, the number of ways $= {}^{3}C_2 \times {}^{17}C_9 = \frac{3!}{2!(3-2)!} = \frac{17!}{9!(17-9)!} = 72\,930$

Example 18

Encik Samad wants to choose three types of batik motifs from four organic motifs and five geometrical motifs. Find the number of ways to choose at least one organic motif and one geometrical motif.

Solution

Number of ways to choose two organic motifs and one geometric motif, ${}^4C_2 \times {}^5C_1$.
Number of ways to choose one organic motif and two geometric motifs, ${}^4C_1 \times {}^5C_2$.
So, the number of ways $= {}^4C_2 \times {}^5C_1 + {}^4C_1 \times {}^5C_2 = 70$.

Self-Exercise 4.8

1. 5 different books will be given to 3 pupils. 2 pupils will get 2 books each while one pupil will get one book. How many ways are there to divide all the books?
2. In one examination, Singham is required to answer two out of three questions from Section A and four out of six questions from Section B. Find the number of ways in which Singham can answer those questions.
3. There are five male graduates and six female graduates who come for interviews at a company. How many ways can the employer select seven employees if
   (a) all the male graduates and two of the female graduates are employed?
   (b) at least five female graduates are employed?

Formative Exercise 4.2

Quiz bit.ly/3jS1nP9

1. By using the formula ${}^nC_r = \frac{n!}{(n-r)!r!}$, show that ${}^nC_r = {}^nC_{n-r}$.
2. A committee of five shall be elected out of five men and three women. Find the number of committees that can be formed if
   (a) there is no condition,
   (b) it contains three men and two women,
   (c) it contains not more than one woman.
3. A team of five members will be selected for an expedition to an island from a group of four swimmers and three non-swimmers. Find the number of ways in which the team can be formed if swimmers must be more than non-swimmers.
4. A mathematics test consists of 10 questions where four of them are questions from trigonometry and six are questions from algebra. Candidates are required to answer only eight questions. Find the number of ways in which a candidate can answer the questions if he answers at least four questions from algebra.
5. A delegation to Malacca consisting of 12 people has been planned. Find the number of ways to provide transport for these 12 passengers if
   (a) three cars are used and each car can accommodate four people,
   (b) two vans are used and each van can accommodate six people.

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REFLECTION CORNER

PERMUTATION AND COMBINATION

Multiplication Rule

If an event can occur in m ways and a second event can occur in n ways, both events can occur in $m \times n$ ways.

Permutation
Order of arrangement is important

- The number of permutations for n different objects is represented by
$$n! = {}^{n}P_{n}$$
- The number of permutations for n different objects when r objects are selected at a time is represented by
$${}^{n}P_{r} = \frac{n!}{(n-r)!}$$

Combination
Order of arrangement is not important

The number of combinations of n different objects when r objects are selected at a time is represented by
$${}^{n}C_{r} = \frac{{}^{n}P_{r}}{r!} = \frac{n!}{r!(n-r)!}$$

Circular Permutations

- Number of permutations for n different objects is represented by
$$P = \frac{n!}{n} = (n-1)!$$
- Number of permutations for n different objects when r objects are selected at a time is represented by
$$P = \frac{{}^{n}P_{r}}{r}$$

Identical Objects

Number of permutations for n objects involving identical objects is represented by
$$P = \frac{n!}{a!b!c!\ldots}$$
where $a, b, c, \ldots$ are the number of identical objects for each type.

Applications

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Journal Writing

1. Construct an infographic on the differences between permutations and combinations.
2. List two problems that occur in your daily life and solve these problems by using the concepts of permutations and combinations that you have learnt.

Summative Exercise

1. Find the number of four-letter codes that can be formed from the letters in the word SEMBUNYI if no letters can be repeated. How many of these codes start with a consonant? PL 2

2. Calculate the probability for someone to guess a password of a laptop containing six characters that are selected from all the numbers and alphabets. PL 3

3. Find the number of ways the letters in the word PULAS can be arranged if each arrangement PL 3
   (a) does not begin with the letter S,
   (b) does not end with S or P.

4. In a futsal match, a match can end with a win, loss or draw. If the Red Eagle Futsal Team joins five futsal matches, find the number of ways in which a match can end up. PL 4

5. Find the number of possible arrangements for the letters in the word JANJANG if the letter N and the letter G must be together.

6. A textile shop sells certain shirts in four sizes, namely S, M, L and XL. If the stocks available in the store consist of two of size S, three of size M, six of size L and two of size XL, find the number of ways to sell all the shirts at the store. PL 3

7. Siew Lin bought seven different young trees to decorate the mini garden at her house. Due to limited space, she can only plant five trees in a circle. Determine the number of ways in which Siew Lin can plant the young trees. PL 3

8. Find the number of ways for six people, namely, Amin, Budi, Cheng, Deepak, Emma and Fakhrul, to sit at a round table if PL 4
   (a) Emma and Fakhrul must sit side by side,
   (b) Emma and Fakhrul cannot sit side by side.

9. 12 stalks of flowers consisting of three red flowers, four blue flowers and five white flowers will be attached onto a string to make a wreath of flowers. Calculate the number of ways to arrange the flowers to make the wreath. PL 3

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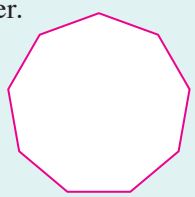
10. An entrance test to a private school contains six questions in Part A and seven questions in Part B. Each candidate needs to answer 10 questions, of which at least four questions are from Part A. Find the number of ways a candidate can answer these 10 questions. PL 5

11. A local community committee of three members are to be selected from four couples. Find the number of ways to select these committee members if PL 4
(a) no condition is imposed,
(b) all members of the committee are husbands,
(c) a husband and his wife cannot serve in the same committee together.

12. A taxi has a seat in the front and three seats at the back. Zara and her three friends wanted to take a taxi, find the number of possible ways where they can choose their seats if PL 4
(a) no condition is imposed,
(b) Zara wants to sit in the front,
(c) Zara wants to sit at the back.

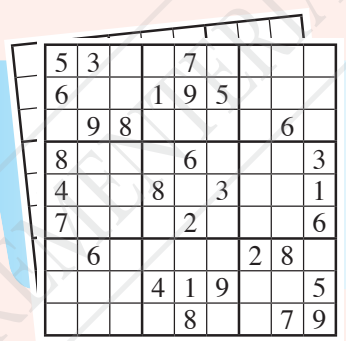
13. There are 15 pupils who enjoy solving puzzles. They meet each other to solve puzzles. At their first meeting, they shake hands with each other. Find the number of handshakes if PL 5
(a) all shake hands with one another,
(b) three people who know one another do not shake hands with each other.

14. Using the vertices of a nonagon, find the number of PL 5
(a) a straight line that can be drawn,
(b) triangles that can be formed,
(c) rectangles that can be formed.



CHAPTER 4

MATHEMATICAL EXPLORATION



5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

SUDOKU

Sudoku is a game based on logic and it involves the placement of numbers. Sudoku was introduced in 1979 but became popular around 2005. The goal of a Sudoku game is to insert one digit between one and nine in one grid cell 9 × 9 with 3 × 3 sub-grids. Each row, column and sub-grid can only be filled by digits from one to nine without repetition.

(a) In your opinion, does this Sudoku game use the concept of permutations or combinations? Explain your answer.
(b) How many ways can you fill in the digits in the first row of a Sudoku game?
(c) How many ways can you solve a Sudoku game?

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CHAPTER 5
PROBABILITY DISTRIBUTION



What will be learnt?

- Random Variable
- Binomial Distribution
- Normal Distribution

List of Learning Standards

bit.ly/3hv5mQd

Malaysia archers created history when her archers representing the country managed to qualify for the Finals in the Asian Cup Archery Championship 2019. In the game, an archer must shoot at least 72 arrows in 12 phases from a 70-metre range. The time given to shoot any three arrows is two minutes while the time given for the last six arrows is four minutes. In your opinion, what are the probabilities for the archers to win? Does each shot depend on the shot before it?



Info Corner

Girolamo Cardano (1501-1576) was the first person to study dice throwing. He had written many books explaining systematically the complete concept of probability.

In the 17th century, two French mathematicians, Blaise Pascal and Pierre de Fermat, formulated the probability theory.

For more info:

bit.ly/32tf54y

Significance of the Chapter

- The knowledge of probability plays an important role in the manufacturing sector.
- This process allows sampling for testing a few samples from thousands of products produced in order to pass quality control and reduce cost.

Key words

Random variable	*Pemboleh ubah rawak*
Discrete random variable	*Pemboleh ubah rawak diskret*
Continuous random variable	*Pemboleh ubah rawak selanjar*
Binomial distribution	*Taburan binomial*
Normal distribution	*Taburan normal*
Mean	*Min*
Variance	*Varians*
Standard deviation	*Sisihan piawai*

Video about Malaysian archers

bit.ly/2PQs0aG

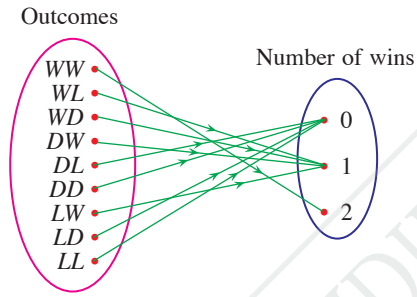
BAHAGIAN PERBUKUAN MALAYSIA

5.1 Random Variable

Random variable

In a basketball competition, the result of any two matches can be recorded as win (W), lose (L) or draw (D). In this case, the sample space can be written as $\{WW, WL, WD, DW, DL, DD, LW, LD, LL\}$. If we only consider the number of wins in the two matches played, then the number of times of winning can be none (0), once (1) or twice (2).

The arrow diagram below shows the relation between all the outcomes of the sample space with the number of wins from the two basketball matches.



Recall

A sample space is a set that consists of all possible outcomes of an experiment.

DISCUSSION

Is the mass of 40 pupils in a class considered as a random variable? Explain.

The numbers 0, 1 and 2 in the arrow diagram represent the number of wins. Set $\{0, 1, 2\}$ is an example of a random variable whose values cannot be determined beforehand and depend on chances.

In general,

> A random variable is a variable with numeric values that can be determined from a random phenomenon.

A random variable can be represented by X and values of the random variable can be represented by r. From the above situation, the random variable X for the number of wins can be written in a set notation, $X = \{0, 1, 2\}$.

Example 1

State the random variable for each of the following situations.
(a) A dice is thrown once.
(b) A man is waiting for a bus at a bus stop.



Solution

(a) The random variable is the number on the top surface of a dice, namely $\{1, 2, 3, 4, 5, 6\}$.
(b) The random variable is the length of time spent at a bus stop.

KEMENTERIAN PENDIDIKAN MALAYSIA

Self-Exercise 5.1

1. State the random variable for each of the following situations in a set notation.
   (a) The result of the Malaysian football team in SEA games.
   (b) The number of white cars among five cars in the parking lot.
   (c) The number of times a head appears when a coin is tossed three times.

2. A ball is taken out of a box which contains a few red and blue balls. After the colour of the ball is recorded, the ball is returned to the box and this process is repeated four times. If X represents the number of times a red ball is chosen from the box, list all the possible outcomes for X in a set notation.



Discrete random variable and continuous random variable

There are two types of random variables to be studied, namely discrete random variables and continuous random variables. A discrete random variable has countable number of values whereas a continuous random variable takes values between a certain interval. Let's explore the differences between these two random variables.

Discovery Activity 1 Group 21st cl

Aim: To compare and contrast discrete random variable and continuous random variable

Steps:

1. Divide the pupils into two groups. The first group will carry out Activity 1 related to discrete random variables. The second group will carry out Activity 2 related to continuous random variables.

Activity 1

1. Get ready a piece of coin.
2. Toss the coin three times in a row.
3. Record whether you get head (H) or tail (T) for each toss.
4. Repeat steps 2 and 3.
5. Then, write all the possible values for the random variable X which represents the number of heads obtained from the three tosses.

Activity 2

1. Measure all the heights (in cm) of the pupils in your class.
2. Record your results on a piece of paper.
3. Then, write the range of the possible values for the random variable Y which represents the heights obtained from the pupils.

2. Next, compare the results obtained by the two groups.
3. What can you deduce from the values of the random variables and the ways they are presented in set notation for the discrete random variable and the continuous random variable? Explain.
4. Present the group findings to the class. Explain the differences between a discrete random variable and a continuous random variable.

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From Discovery Activity 1 results, it is found that:

- Random variables that have countable numbers of values, usually taking values like zero and positive integers, are known as **discrete random variables**.
- Random variables that are not integers but take values that lie in an interval are known as **continuous random variables**.

If X represents a discrete random variable, hence the possible outcomes can be written in set notation, $X = \{r : r = 0, 1, 2, 3\}$.

If Y represents a continuous random variable, hence the possible outcomes can be written as $Y = \{y : y$ is the pupil's height in cm, $a \leqslant y \leqslant b\}$.

Example 2

Write down all the possible outcomes in set notations for each of the following events. Determine whether the event is a discrete random variable or a continuous random variable. Explain.

(a) A fair dice is thrown three times, given X is a random variable which represents the number of times to get the number 4.

(b) X is a random variable which represents the time taken by a pupil to wait for his bus at a bus stop. The range of time taken by the pupil is between 5 to 55 minutes.

Solution

(a) $X = \{0, 1, 2, 3\}$. The event is a discrete random variable because its values can be counted.

(b) $X = \{x : x$ is the time in minutes where $5 \leqslant x \leqslant 55\}$. The event is a continuous random variable because its values lie in an interval from 5 to 55 minutes.

Self-Exercise 5.2

1. Write down all the possible outcomes in set notations for each of the following events. Determine whether the event is a discrete random variable or a continuous random variable.

(a) Six prefects are randomly selected from pupils of Form 5. X represents the number of prefects who wear glasses.

(b) Seven patients are randomly selected from a hospital for blood tests. X represents the number of unprivileged patients.

(c) The shortest building in Seroja city is 3 m while the tallest is 460 m. X represents the heights of the buildings located in the city of Seroja.

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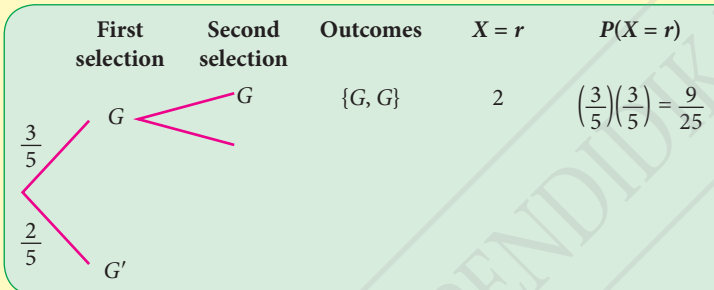
Probability distribution for discrete random variables

Discovery Activity 2 Pair

Aim: To describe the meaning of a probability distribution for a discrete random variable X by using a tree diagram

Steps:

1. Prepare five pieces of square paper and write a number taken from 1 to 5 on each paper.
2. Put the five pieces of paper in a small box.
3. Take a piece of the paper from the box at random and record the number obtained. Return the paper into the box before choosing another piece. This process is repeated twice.
4. If X is the number of times of getting an odd number, write
   (a) all the possible values of X in the two selections,
   (b) the probability of selecting an odd number each time.
5. Then, complete the following tree diagram.



6. From the tree diagram, find
   (a) the probability for each value of X,
   (b) the total probability.
7. Draw a conclusion on the probability for each value of the random variable X and the total probability of the distribution.

Recall

- Probability of an event A occurring, $P(A) = \frac{n(A)}{n(S)}$ where $n(A)$ is the number of outcomes for the event A and $n(S)$ is the number of outcomes in the sample space S.
- Probability of an event A occurring is from 0 to 1, that is, $0 \leqslant P(A) \leqslant 1$.
- If the event A' is the complement of the event A, then $P(A') = 1 - P(A)$.

From Discovery Activity 2, it shows that the possible values for X are 0, 1 and 2. Each of these numbers represents the events from the sample space $\{(G, G), (G, G'), (G', G), (G', G')\}$. The probability of each event can be summarised in the probability distribution table for X as shown in the table on the side. In general,

$X = r$	0	1	2
$P(X = r)$	$\frac{4}{25}$	$\frac{12}{25}$	$\frac{9}{25}$

If X is a discrete random variable with the values $r_1, r_2, r_3, \ldots, r_n$ and their respective probabilities are $P(X = r_1), P(X = r_2), P(X = r_3), \ldots, P(X = r_n)$, then $\sum_{i=1}^{n} P(X = r_i) = 1$, thus each $P(X = r_i) \geqslant 0$.

Example 3



Two fair dice are tossed together three times. Let X be a discrete random variable for getting 7 from the sum of the numbers on the two dice.

(a) Write the values of X in a set notation.

(b) Draw a tree diagram to represent all the possible outcomes of X.

(c) From the tree diagram in (b), find the probability for each possible value of X.

(d) Determine the total probability for the distribution of X.

Solution

(a) $X = \{0, 1, 2, 3\}$

(b) Let R be the results of getting 7 and T be those results of not getting 7.

First dice

Second dice +	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

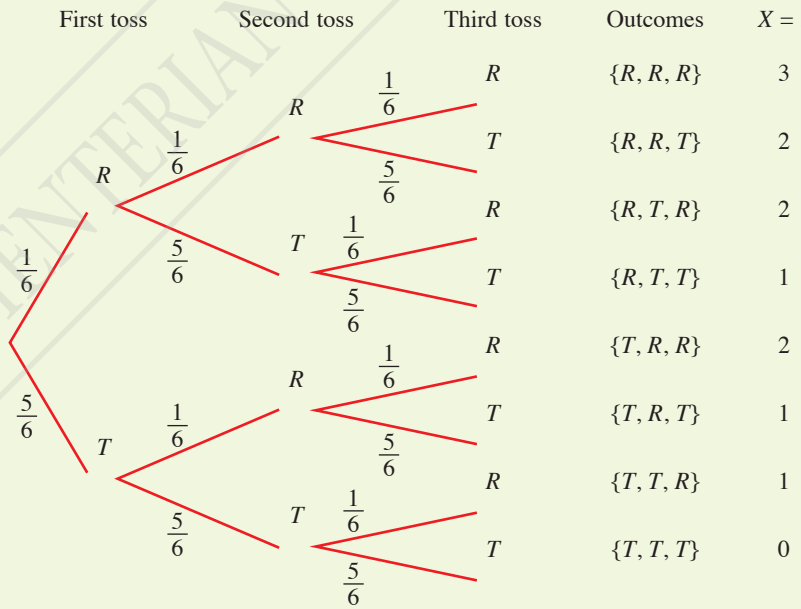
Excellent Tip

By using the multiplication rule,

$${}^6C_1 \times {}^6C_1 = 36$$

Thus, the number of outcomes in the sample space, $n(S)$ for Example 3 is 36.

From the above table, the probability of getting 7 in each trial is $\frac{6}{36} = \frac{1}{6}$.



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(c) $P(X = 0)$
$= P(T, T, T)$
$= \frac{5}{6} \times \frac{5}{6} \times \frac{5}{6}$
$= \frac{125}{216}$
$= 0.5787$

$P(X = 2)$
$= P(R, R, T) + P(R, T, R) + P(T, R, R)$
$= \left(\frac{1}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{1}{6} \times \frac{5}{6} \times \frac{1}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{1}{6}\right)$
$= \frac{15}{216}$
$= 0.0695$

$P(X = 3)$
$= P(R, R, R)$
$= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6}$
$= \frac{1}{216}$
$= 0.0046$

$P(X = 1)$
$= P(R, T, T) + P(T, R, T) + P(T, T, R)$
$= \left(\frac{1}{6} \times \frac{5}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{1}{6} \times \frac{5}{6}\right) + \left(\frac{5}{6} \times \frac{5}{6} \times \frac{1}{6}\right)$
$= \frac{75}{216}$
$= 0.3472$

(d) The total probability $= \frac{125}{216} + \frac{75}{216} + \frac{15}{216} + \frac{1}{216}$
$= 1$

Excellent Tip

In Example 3,

$P(X = 1) = {}^3C_1\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^2$
$= 0.3472$

$P(X = 2) = {}^3C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^1$
$= 0.0695$

CHAPTER 5

Self-Exercise 5.3

1. In a mini hall, there are three switches to turn on three fans. X represents the number of switches that are turned on at a time.
(a) Write X in a set notation.
(b) Draw a tree diagram to show all the possible outcomes and find the probability for each of them.
(c) Determine the total probability distribution of X.



2. In 2016, it was found that 38% of the cars purchased by Malaysians were white. If two buyers were selected at random and X represents the number of white car's buyers,
(a) state the set of X,
(b) draw a tree diagram and determine the probability distribution of X.

3. A coin is tossed three times and X represents the number of times of getting 'heads'.
(a) Write X in a set notation.
(b) Draw a tree diagram to represent all the possible outcomes of X.
(c) Show that X is a discrete random variable.

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Table and graph of probability distribution for discrete random variable

In addition to the tree diagram, the probability distribution for each discrete random variable X can be represented by a table and a graph. The table as well as the graph can display the values of the discrete random variable with their corresponding probabilities.

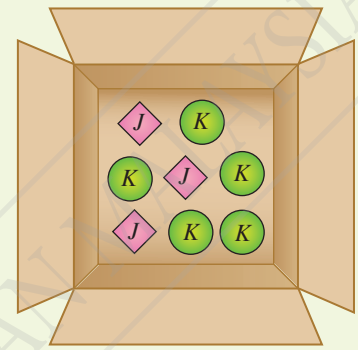
Example 4

In a factory, a supervisor wants to check the quality of a certain product at random. There are 3 type-J products and 5 type-K products in a box. The supervisor will randomly pick one product and the product type will be recorded. The product will then be returned to the box and the process is repeated three times. Let X represent the number of times type-K product is inspected.

(a) Write X in a set notation.

(b) Draw a tree diagram to represent all the possible outcomes of X.

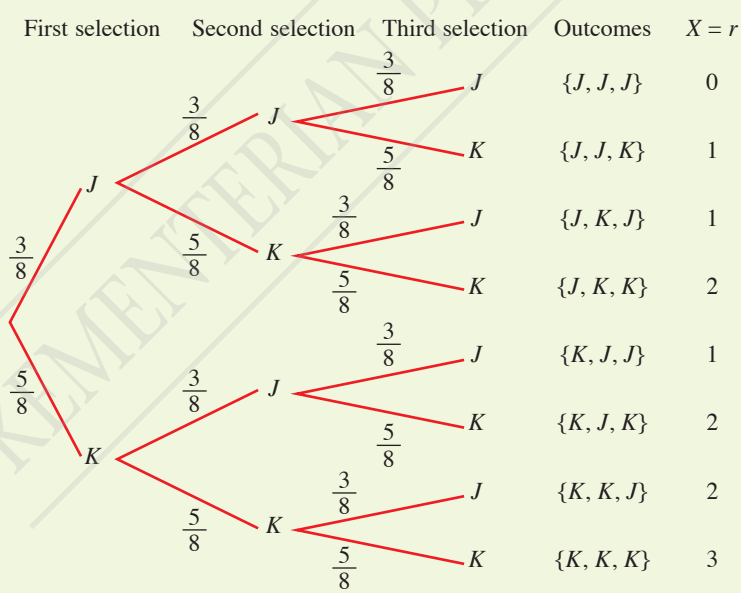
(c) List the distribution of the values of X together with their respective probabilities in a table and then draw a graph to show the probability distribution of X.



Solution

(a) $X = \{0, 1, 2, 3\}$

(b)



Excellent Tip

The choice of the second or the third product is not dependent on the choice of the first product as the earlier product has been returned to the box. These are independent events.

Flash Quiz

If the first product selected is not returned to the box, is the probability of getting the next product still the same? If not, find the probabilities of getting the second and the third type-K products.

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(c) $P(X = 0)$
$= P(J, J, J)$
$= \frac{3}{8} \times \frac{3}{8} \times \frac{3}{8}$
$= \frac{27}{512}$
$= 0.0527$

$P(X = 1)$
$= P(J, J, K) + P(J, K, J) + P(K, J, J)$
$= \left(\frac{3}{8} \times \frac{3}{8} \times \frac{5}{8}\right) + \left(\frac{3}{8} \times \frac{5}{8} \times \frac{3}{8}\right) + \left(\frac{5}{8} \times \frac{3}{8} \times \frac{3}{8}\right)$
$= \frac{135}{512}$
$= 0.2637$

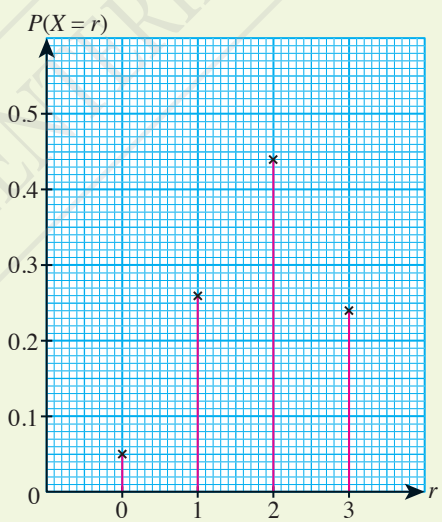
$P(X = 2)$
$= P(J, K, K) + P(K, J, K) + P(K, K, J)$
$= \left(\frac{3}{8} \times \frac{5}{8} \times \frac{5}{8}\right) + \left(\frac{5}{8} \times \frac{3}{8} \times \frac{5}{8}\right) + \left(\frac{5}{8} \times \frac{5}{8} \times \frac{3}{8}\right)$
$= \frac{225}{512}$
$= 0.4395$

$P(X = 3)$
$= P(K, K, K)$
$= \frac{5}{8} \times \frac{5}{8} \times \frac{5}{8}$
$= \frac{125}{512}$
$= 0.2441$

Presenting the probability distribution of X in a table:

$X = r$	0	1	2	3
$P(X = r)$	0.0527	0.2637	0.4395	0.2441

Presenting the probability distribution of X in a graph of $P(X = r)$ against r:



Alternative Method

For $P(X = 1)$, the choice of getting type-K product once can happen during the first, second or third selection. Hence, the concept of combination can be applied.

$${}^3C_1\left(\frac{5}{8}\right)^1\left(\frac{3}{8}\right)^2 = 3\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)^2$$
$$= \frac{135}{512}$$
$$= 0.2637$$

Flash Quiz

Using the concept of combination, find
(a) $P(X = 0)$
(b) $P(X = 2)$
(c) $P(X = 3)$

Flash Quiz

From the table and the graph in Example 4, what is the total probability distribution of X?

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Example 5

70% of Form 5 Dahlia pupils achieved a grade A in the final year examination for the science subject. Two pupils were chosen at random from that class. If X represents the number of pupils who did not get a grade A, construct a table to show all the possible values of X with their corresponding probabilities. Next, draw a graph to show the probability distribution of X.

Solution

$P(A : A \text{ is a pupil who did not achieve a grade A}) = 1 - \frac{70}{100} = 0.3$

$P(B : B \text{ is a pupil who achieved a grade A}) = \frac{70}{100} = 0.7$

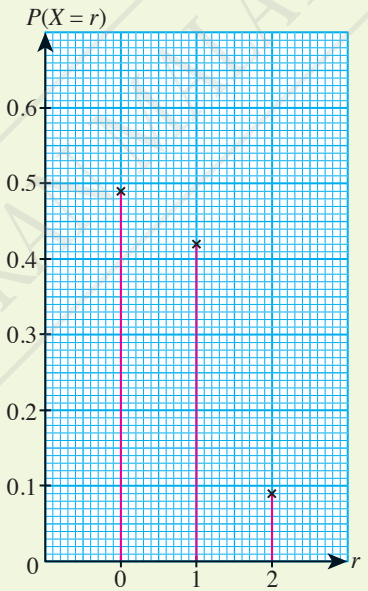
Then, $X = \{0, 1, 2\}$

$P(X = 0) = P(B, B)$
$= 0.7 \times 0.7$
$= 0.49$

$P(X = 1) = P(A, B) + P(B, A)$
$= (0.3 \times 0.7) + (0.7 \times 0.3)$
$= 0.42$

$P(X = 2) = P(A, A)$
$= 0.3 \times 0.3$
$= 0.09$

$X = r$	0	1	2
$P(X = r)$	0.49	0.42	0.09



Self-Exercise 5.4

1. 6 out of 10 pupils randomly selected had attended a leadership camp. If 5 people are selected randomly from that group of pupils and X represents the number of pupils who had participated in the leadership camp, draw a graph to represent the probability distribution of X.

2. It is found that 59% of the candidates who sat for the entrance examination to enter a boarding school passed all the subjects. It is given that 4 pupils are randomly selected from the candidates and X represents the number of pupils who passed all their subjects.
   (a) Construct a probability distribution table for X.
   (b) Then, draw the probability distribution graph for X.

3. There are 2 basketballs and 4 footballs in a box. 4 balls are randomly drawn from the box one at a time. After the type of ball is recorded, it is returned to the box. If X represents the number of basketballs being drawn from the box, draw a probability distribution graph for X.

KEMENTERIAN PENDIDIKAN MALAYSIA

Formative Exercise 5.1

Quiz bit.ly/3aP0xyV

1. A school debate team consists of 6 people, 2 of them are boys. 2 members of the debate team are randomly selected to participate in a contest and X represents the number of boys being selected.
(a) List all the possible values of X.
(b) State whether X is a discrete random variable or a continuous random variable.

2. It is found that the longest nail produced by a factory is 10.2 cm and the shortest nail is 1.2 cm. If X represents the random variable for the lengths of nails produced by the factory,
(a) list all the possible values of X,
(b) state whether X is a discrete random variable or a continuous random variable.



3. Given $X = \{0, 1, 2, 3\}$ is a discrete random variable that represents the number of computers in an office together with their respective probability functions as shown in the table below.

$X = r$	0	1	2	3
$P(X = r)$	0.2	0.35	0.3	0.15

(a) Show that X is a discrete random variable with the probability function $P(X = r)$.
(b) Draw the probability distribution graph for X.

4. A box contains several table tennis balls. Each table tennis ball is labelled with a number taken from 1 to 10. The probability of selecting 1, 3 or 5 is 0.2 while the probability of selecting 2, 4, 6 or 8 is 0.1. A table tennis ball is randomly drawn from the box and it is returned to the box after the digit is recorded. This process is repeated 3 times. If X represents the number of times 1, 3 or 5 are selected,
(a) list all the possible values of the random variable X,
(b) show that X is a discrete random variable with the probability function $P(X = r)$,
(c) draw the probability distribution graph for X.

5. Given $X = \{0, 1, 2, 3, 4\}$ is a discrete random variable with the probability given in the table below.

$X = r$	0	1	2	3	4
$P(X = r)$	p	p	$p + q$	q	q

If $p = 2q$, find the values of p and q.

6. A player will be awarded 1 point if he wins in a chess game. $\frac{1}{2}$ point is given if he gets a draw and 0 point if he loses the game. Lee played three sets of chess games.
(a) Construct a tree diagram to represent all the possible outcomes.
(b) If X represents the number of points obtained by Lee, list the set of X.
(c) Draw a graph of the probability distribution of X.

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5.2 Binomial Distribution

Binomial distribution

Consider the following situations:

> When a fair coin is tossed once, the outcome is either a head or a tail.

Note that the above situation has only two possible outcomes, that is, either getting a head or getting a tail. If the outcome of getting a head is regarded as a 'success', then the outcome of getting a tail will be regarded as a 'failure'. An experiment that produces only two possible outcomes is known as a **Bernoulli trial**. The characteristics of Bernoulli trials are as follows:

- There are only two possible outcomes, namely 'success' and 'failure'.
- The chances of 'success' are always the same in every trial.
- If the probability of 'success' is given by p, then the probability of 'failure' is given by $(1 - p)$ where $0 < p < 1$.
- The discrete random variable $X = \{0, 1\}$, where 0 represents 'failure' and 1 represents 'success'.

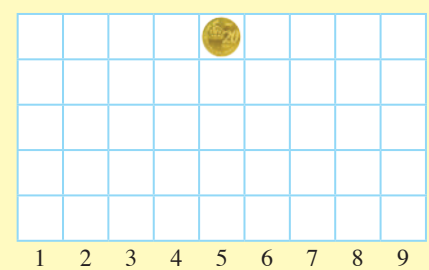
An experiment which is made up of n similar Bernoulli trials is known as a binomial experiment. Let's explore the relation between Bernoulli trials and binomial distribution.

Discovery Activity 3 Group

Aim: To explore the relationship between Bernoulli trials and binomial distribution

Steps:

1. Prepare a piece of display sheet, a fair dice and a fair coin.
2. Draw a grid consisting of five rows and nine columns as shown in the diagram.
3. Place the coin in the square on the first row and fifth column of the grid paper.
4. Toss the dice once and move the coin according to the following instructions:



> - If an odd number appears, move the coin one step down and then one step to the left.
> - If an even number appears, move the coin one step down and then one step to the right.

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5. Toss the dice four times so that the coin moves until it reaches the fifth row.
6. Then, answer the following questions.
   (a) Does the tossing of a dice resemble a Bernoulli trial?
   (b) What is the relation between each toss of the dice? Is the tossing dependent on one another?
   (c) How many types of outcome can be obtained from each toss? List all of them.
   (d) If the discrete random variable X represents the number of times of getting an even number from each toss of the dice, write the values of X in a set notation.

From Discovery Activity 3 results, it is noted that:

- The experiment consists of four similar Bernoulli trials.
- Each trial has only two outcomes, which are 'success' and 'failure'.
- The probability of 'success' for each trial is unchanged.
- Each trial is independent, that is, the earlier outcome does not affect the subsequent outcomes.

The above mentioned characteristics are known as a **binomial experiment**. In general,

A binomial random variable is the number of success r from n similar Bernoulli trials in a binomial experiment. The probability distribution of a binomial random variable is known as a **binomial distribution**.

HISTORY GALLERY

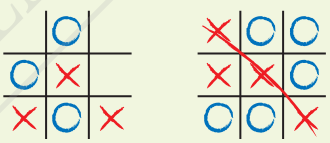


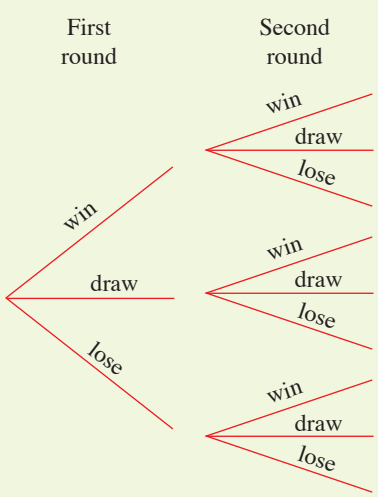
Jacob Bernoulli was a 17th century Swiss mathematician. He studied the characteristics of trials whose 'success' outcomes had the same probabilities when the trials were repeated.

CHAPTER 5

Example 6

The diagram on the right shows a tree diagram of all the possible outcomes after two rounds of tic-tac-toe game. Is this a binomial distribution? Explain.





Solution

This distribution has three possible outcomes, namely win, draw or lose. Therefore, this distribution is not a binomial distribution because a binomial distribution has only two possible outcomes for each trial.

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Example 7

A shelf contains 6 identical copies of chemistry reference books and 4 identical copies of physics reference books. 3 copies of the physics reference books are taken at random from the shelf one after another without replacement. State whether this probability distribution is a binomial distribution or not. Explain.

Solution

$P(\text{getting the 1st copy of physics reference book}) = \frac{4}{10} = \frac{2}{5}$

$P(\text{getting the 2nd copy of physics reference book}) = \frac{3}{9} = \frac{1}{3}$

$P(\text{getting the 3rd copy of physics reference book}) = \frac{2}{8} = \frac{1}{4}$

Excellent Tip

An experiment with n equals to 1 is a Bernoulli trial.

The probability of getting a copy of the physics reference book in each trial changes and each outcome depends on the previous outcome.

Thus, the probability distribution of getting 3 copies of physics reference books without replacement is not a binomial distribution.

Self-Exercise 5.5

1. Given X is a discrete random variable of a Bernoulli trial with the probability of 'success' being 0.3.
(a) List all the elements in set X.
(b) Find the probability of 'failure'.

2. An experiment was conducted by tossing a 50 cent coin on the first trial and then tossing a dice on the second trial. Explain whether this experiment is a binomial experiment or not.

3. An association conducted a survey on the monthly wage earned by most of the working-class Malaysians. The result of the survey showed that 50% of the working-class Malaysians earn less than RM2 000 a month. If 3 workers are randomly selected from a group of workers, explain whether the probability distribution is a binomial distribution or not.

4. In a survey, it is found that 9 out of 10 students from a certain college have part-time jobs. If 4 students are randomly selected from that college, is the probability distribution for students doing part-time jobs binomially distributed? Explain.

5. It is found that a SPM graduate student has three options, namely; continues his studies locally, continues his studies abroad or stops studying. A student is randomly selected from this group of students. Draw a tree diagram to show all the possible outcomes. Explain whether the outcomes have the characteristics of a binomial distribution.

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Probability of an event for binomial distribution

If a binomial random variable X represents the number of 'success' in n independent trials of an experiment, with p as the probability of 'success' and $q = 1 - p$ as the probability of 'failure', then the binomial probability function for X is given by the following formula:

$$P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r},\ r = 1, 2, 3, \ldots, n$$

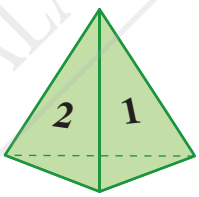
We can also write it as $X \sim \mathrm{B}(n, p)$.

Excellent Tip

The event of getting a success or a failure is a mutually exclusive event.

Consider the following event:

A triangular pyramid with four flat surfaces of equal size are labelled with a number from 1 to 4. Naim flips the triangular pyramid 3 times. What is the probability of the pyramid sitting on number 4 after each flip?



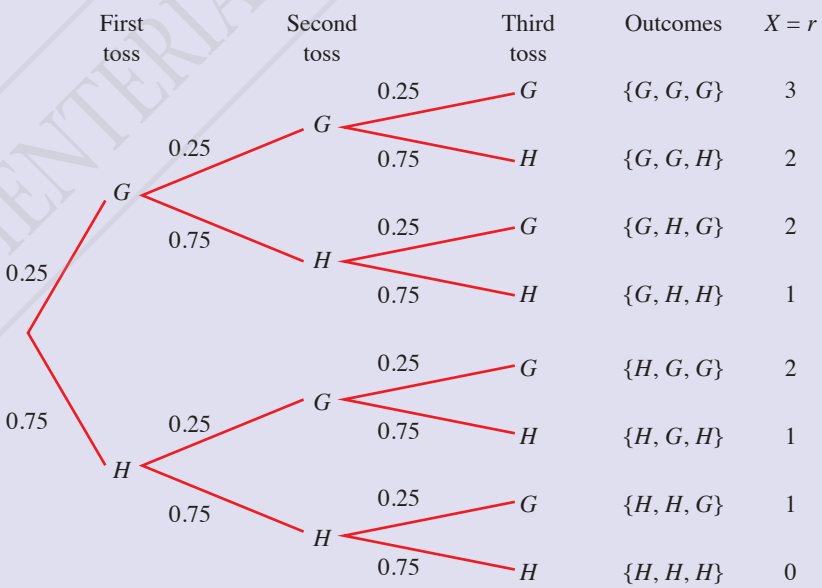
Note that flipping a triangular pyramid 3 times is a binomial experiment with $n = 3$. So, the probability of the pyramid sitting on number 4 after each flip is:

$$p = \frac{1}{4} = 0.25 \qquad \text{and} \qquad q = (1 - p) = \frac{3}{4} = 0.75$$

If X represents a random variable for the number of times the pyramid sits on number 4, then $X = \{0, 1, 2, 3\}$.

Let G = the outcome of the pyramid sitting on number 4
and H = the outcome of the pyramid not sitting on number 4

All the possible outcomes of the triangular pyramid after every flip can be shown in the tree diagram below.



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The table below shows all the results and distributions of their respective probabilities based on the tree diagram and on the binomial distribution formula.

	From the tree diagram	From the binomial distribution formula
$X = r$	$P(X = r)$	$P(X = r)$
0	$P(X = 0) = P(H, H, H)$ $= 0.75^3$ $= 0.4219$	${}^3C_0(0.25)^0(0.75)^3 = 0.4219$
1	$P(X = 1) = P(G, H, H) + P(H, G, H) + P(H, H, G)$ $= 3(0.75)^2(0.25)$ $= 0.4219$	${}^3C_1(0.25)^1(0.75)^2 = 0.4219$
2	$P(X = 2) = P(G, G, H) + P(G, H, G) + P(H, G, G)$ $= 3(0.75)(0.25)^2$ $= 0.1406$	${}^3C_2(0.25)^2(0.75)^1 = 0.1406$
3	$P(X = 3) = P(G, G, G)$ $= (0.25)^3$ $= 0.0156$	${}^3C_3(0.25)^3(0.75)^0 = 0.0156$

Note that the two methods, namely using a tree diagram and using the binomial distribution formula yield the same probability values for each of the values of the binomial random variable X. However, the tree diagram will be difficult to draw once the number of flips exceeds three.

The probability of the pyramid sitting on number 4 less than 2 times,

$P(X < 2) = P(X = 0) + P(X = 1)$
$= 0.4219 + 0.4219$
$= 0.8438$

The probability of the pyramid sitting on number 4 more than 0 times,

$P(X > 0) = P(X = 1) + P(X = 2) + P(X = 3)$
$= 1 - P(X = 0)$
$= 1 - 0.4219$
$= 0.5781$

From the table above, the total probability for the random variable X is:

$P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$
$= 0.4219 + 0.4219 + 0.1406 + 0.0156$
$= 1$

In general,

$$\sum_{i=1}^{n} P(X = r_i) = 1$$

QR Access

Prove that $\sum_{i=1}^{n} P(X = r_i) = 1$

bit.ly/2ErN1oI

Flash Quiz

What is the probability of the pyramid sitting on number 4 less than once or more than twice?

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Example 8

The probability that it rains on a certain day is 0.45. By using the formula, find the probability that in a particular week, it rains
(a) exactly 4 days,
(b) at least 2 days.

Solution

Let X represent the number of rainy days.
Given $n = 7$, $p = 0.45$ and $q = 0.55$,

(a) $P(X = 4) = {}^7C_4(0.45)^4(0.55)^3$
$= 0.2388$

(b) $P(X \geqslant 2) = P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6) + P(X = 7)$
$= 1 - [P(X = 0) + P(X = 1)]$
$= 1 - [{}^7C_0(0.45)^0(0.55)^7 + {}^7C_1(0.45)^1(0.55)^6]$
$= 1 - 0.0152 - 0.0872$
$= 0.8976$

Excellent Tip

${}^nC_r$ means that there are r 'success' in n trials. Based on Example 8(a), in 7 days, any 4 days are chosen.

Choose 4 out of 7
↓
${}^7C_4(0.45)^4(0.55)^3$
↑ 4 times the probability of 'success'
↑ 3 times the probability of 'failure'

CHAPTER 5

Self-Exercise 5.6

1. In 2019, the estimated population of Malaysia was 32.6 million people. In one of the surveys, it was found that about 57% of Malaysians use smartphones. A sample of 8 people was selected at random. Find the probability that
(a) 6 of them use smartphones,
(b) not more than 2 of them use smartphones.

2. On a shelf, there are 3 novels and 2 comic books. A book is chosen from the shelf and after reading, it is returned before the next book is chosen from the shelf. This process is repeated 3 times. If X represents the random variable of choosing a comic book from the shelf,
(a) construct a tree diagram to show all the possible outcomes,
(b) find the probability of choosing
(i) a comic book only once,
(ii) a novel three times.

3. In a survey, it is found that 95% of undergraduates at a university own laptops. A sample consisting of 8 undergraduates is selected at random from the university. Find the probability that
(a) exactly 6 of them have laptops,
(b) at most 2 or more than 7 of them own laptops.

4. Given a discrete random variable $X \sim \text{B}(n, 0.65)$,
(a) find the value of n if $P(X = n) = 0.0319$,
(b) based on the answer in (a), find $P(X > 2)$.

KEMENTERIAN PENDIDIKAN MALAYSIA

Constructing table, drawing graph and interpreting information of binomial distribution

Discovery Activity 4

Group | 21st cl | STEM | CT

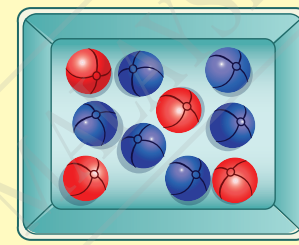
Aim: To construct tables, draw graphs and interpret information from the binomial distribution

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Steps:

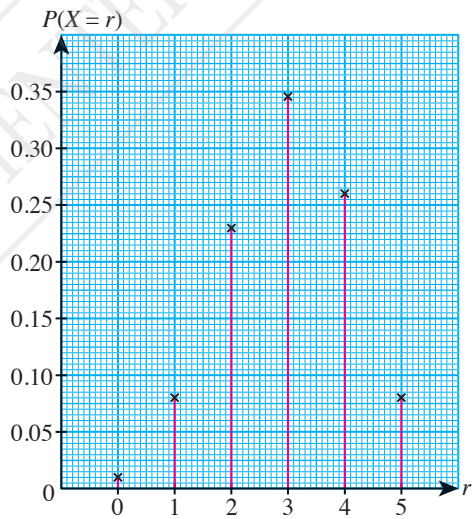
1. Form a few groups, each with four members.

   - Prepare a container. Put 4 red balls and 6 blue balls into the container.
   - One of the group members will choose a ball from the container randomly.
   - Others in the group will record the colour of the ball being chosen and then the ball is returned to the box.
   - This process is repeated five times.



2. Suppose X is the random variable of choosing a blue ball, by using the formula $P(X = r) = {}^{n}C_{r}\, p^{r} q^{n-r}$, where $r = 0, 1, 2, 3, 4, 5$. Construct a probability distribution table.
3. Then, construct a probability distribution graph by using a dynamic geometry software called GeoGebra by scanning the QR code or browsing the provided link above.
4. From the probability distribution table and the graph drawn, find the following probabilities.
   (a) $P(X = 3)$, (b) $P(X < 3)$, (c) $P(1 < X < 3)$.
5. How do you determine the probabilities from the table and the graph?
6. Present your group's results to the class.

From Discovery Activity 4, it is found that the probability of the random variable X of a binomial distribution can be obtained from the table as well as from the probability distribution graph. The probability distribution graph can be drawn as shown in the diagram below.



Excellent Tip

For any n of a binomial distribution:

- When $p = 0.5$, the graph is symmetrical.
- When $p < 0.5$, the graph is skewed to the left and is not symmetrical.
- When $p > 0.5$, the graph is skewed to the right and is not symmetrical.

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Example 9

Emma did a survey on the percentage of pupils in her school who use school buses to come to school. It is found that 45% of pupils from her school use school buses. A sample of 4 pupils is randomly selected from the school.

(a) Construct a binomial probability distribution table for the number of pupils who use school buses.

(b) Draw a graph for this distribution.

(c) From the table or graph, find the probability that

(i) more than 3 pupils come to school by school buses,

(ii) less than 2 pupils use school buses.

Solution

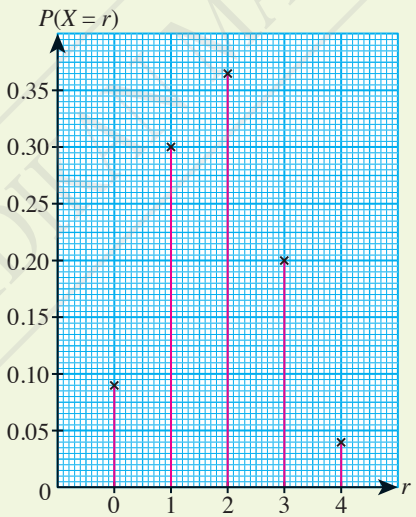
(a) Let X represent the number of pupils who use school buses.

Then, $X = \{0, 1, 2, 3, 4\}$.

Given $n = 4$, $p = 0.45$ and $q = 0.55$

$X = r$	$P(X = r)$
0	${}^4C_0(0.45)^0(0.55)^4 = 0.0915$
1	${}^4C_1(0.45)^1(0.55)^3 = 0.2995$
2	${}^4C_2(0.45)^2(0.55)^2 = 0.3675$
3	${}^4C_3(0.45)^3(0.55)^1 = 0.2005$
4	${}^4C_4(0.45)^4(0.55)^0 = 0.0410$

(b)



(c) (i) $P(X > 3) = P(X = 4)$
$= 0.0410$

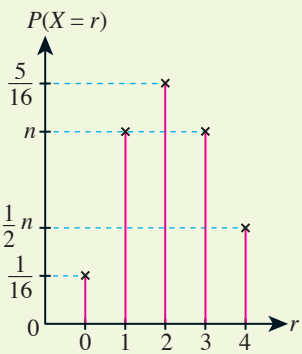
(ii) $P(X < 2) = P(X = 0) + P(X = 1)$
$= 0.0915 + 0.2995$
$= 0.3910$

Example 10

The diagram on the right shows a binomial distribution graph.

(a) State all the possible outcomes of X.

(b) Find the value of n.



Solution

(a) $X = \{0, 1, 2, 3, 4\}$

(b) $P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 1$

$$\frac{1}{16} + \frac{1}{2}n + n + \frac{5}{16} + n = 1$$

$$n = \frac{1}{4}$$

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Self-Exercise 5.7

1. It is found that 35% of Form 5 Bestari pupils achieved a grade B in additional mathematics. If 6 pupils are randomly selected from that class, find the probability that
(a) 4 pupils achieved a grade B,
(b) more than one pupil achieved a grade B.

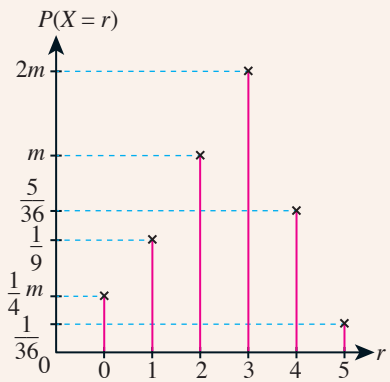
2. In a study, the probability that a certain type of smartphone is spoilt after 3 years is 78%.
(a) If 7 of these smartphones are randomly chosen, find the probability that 4 of them are spoilt after 3 years.
(b) Find the number of smartphones that are spoilt if the sample is 200.

3. In one report, 54% of Malaysians buy locally made cars. If 8 people who just bought new cars are selected at random, find the probability that
(a) at least 2 of them bought locally made cars,
(b) more than 6 of them bought locally made cars.



4. It is found that the probability of an electronic factory to produce faulty printing machines is 0.05. Five printing machines are randomly chosen from the factory.
(a) Construct a probability distribution table for the number of faulty printing machines and then draw a graph.
(b) From the table or graph, find the probability that
(i) exactly 2 printing machines are faulty,
(ii) more than one printing machines are faulty.

5. The diagram on the right shows a binomial distribution graph for the discrete random variable X.
(a) State all the possible outcomes of X.
(b) Find the value of m from the graph.
(c) Find the percentage for $P(X \geqslant 2)$.



6. In a study, it is found that 17% of Malaysians aged 18 years and above have diabetes. If 10 people are randomly selected from that age group, find
(a) the probability that 5 of them have diabetes,
(b) $P(2 \leqslant X \leqslant 6)$ where X represents the number of Malaysian citizens aged 18 and above who have diabetes.

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The value of mean, variance and standard deviation for a binomial distribution

You have learnt that a binomial distribution is made up of n independent Bernoulli trials and each trial has the same probability of 'success'. What is the mean or expected value of this binomial distribution? Let's explore.

Discovery Activity 5 Pair 21st cl

Aim: Determine the mean value of a binomial distribution

Steps:

1. Consider the two situations below.

Situation 1

A fair coin is tossed 100 times. The variable X represents the number of times the heads is obtained.

Situation 2

A test consists of 60 multiple choice questions where each question has four choices. A pupil answers all the questions randomly. The variable X represents the number of questions the pupil answers correctly.

2. From Situation 1, estimate the number of times the heads are obtained, based on the ratio concept. Explain.
3. From Situation 2, estimate the number of questions that are answered correctly based on the ratio concept. Explain.
4. Discuss the answers you get with other pairs.

From Discovery Activity 5, it is found that the expected value of a binomial distribution is the product of the number of trials with its probability of 'success'.

If a discrete random variable X has a binomial distribution, that is, $X \sim \mathrm{B}(n, p)$, then the **expected value** or **mean**, μ of this distribution is defined as the sum of the product of the value of X with its respective probability divided by the total probability of the distribution.

$$\mu = \frac{\sum_{r=0}^{n} r\,P(X = r)}{\sum_{r=0}^{n} P(X = r)}$$

Since $\sum_{r=0}^{n} P(X = r) = 1$, the formula for mean can be summarised as follows.

$$\text{Mean, } \mu = np$$

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A standard deviation, σ is a measure of deviation of a set of data from its mean value.

Variance, σ^2 and the standard deviation, σ for a binomial distribution is given by the following formula.

$$\text{Variance, } \sigma^2 = npq$$

$$\text{Standard deviation, } \sigma = \sqrt{npq}$$

QR Access

Prove the formula of the mean and the variance of a binomial distribution

bit.ly/2QkDlyY

Example 11

A study shows that 95% of Malaysians aged 20 and above have a driving license. If 160 people are randomly selected from this age group, estimate the number of Malaysians aged 20 and above who have a driving license. Then, find the variance and the standard deviation of the distribution.

Solution

Given $p = 0.95$, $q = 0.05$ and $n = 160$

Mean, $\mu = np$

$\mu = 160 \times 0.95$

$\mu = 152$

Variance, $\sigma^2 = npq$

$\sigma^2 = 160 \times 0.95 \times 0.05$

$\sigma^2 = 7.60$

Standard deviation, $\sigma = \sqrt{npq}$

$\sigma = \sqrt{7.6}$

$\sigma = 2.76$

Flash Quiz

Why is standard deviation the square root of the variance? Explain.

Self-Exercise 5.8

1. A discrete random variable X has a binomial distribution, which is $X \sim B(n, p)$ with a mean of 45 and a standard deviation of 3. Find the values of n and p.
2. A discrete random variable $X \sim B(120, 0.4)$. Find its mean and standard deviation.
3. There are 5 000 people in a village. It is found that 8 out of 10 of the villagers installed broadband at home. Find the mean, variance and standard deviation for the number of people who have broadband at home.
4. In a study, it is found that 3 out of 5 men enjoy watching football games. If 1 000 men are randomly selected, find the mean and the standard deviation for the number of men who enjoy watching football games.

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Solving problems involving binomial distributions

Example 12

A cake shop produces a certain chocolate cake. It is found that 12% of the chocolate cake have masses less than 1 kg. Find the minimum number of chocolate cakes that need to be checked if the probability of choosing at random a chocolate cake with a mass less than 1 kg is at least greater than 0.85.



Solution

Let X represent the number of chocolate cakes with masses less than 1 kg.
Then, $X \sim \mathrm{B}(n, p)$ with $p = 0.12$ and $q = 0.88$.

$$P(X \geqslant 1) > 0.85$$
$$1 - P(X = 0) > 0.85$$
$$P(X = 0) < 1 - 0.85$$
$${}^{n}C_{0}(0.12)^{0}(0.88)^{n} < 0.15$$
$$(0.88)^{n} < 0.15$$
$$n \log 0.88 < \log 0.15$$ ← Take log on both sides
$$n > \frac{\log 0.15}{\log 0.88}$$
$$n > 14.84$$

Therefore, the minimum number of cakes to be checked is $n = 15$.

Flash Quiz

In Example 12, state your reason why $n > \dfrac{\log 0.15}{\log 0.88}$ is not $n < \dfrac{\log 0.15}{\log 0.88}$

Example 13 MATHEMATICAL APPLICATIONS

In a survey, 35% of Malaysians born between 1980 to 2000 can afford to own a house. If 10 people are chosen from this group of Malaysians, find the probability that not more than two people can afford to own a house.

Solution

1 . Understanding the problem

- This problem shows binomial characteristics with $n = 10$ and $p = 0.35$.
- Find P(not more than two people can afford to own a house).

2 . Planning the strategy

- Let X represent the number of Malaysians born between 1980 and 2000 who can afford to own a house.
- $P(X \leqslant 2) = P(X = 0) + P(X = 1) + P(X = 2)$ by using the formula $P(X = r) = {}^{n}C_{r}p^{r}q^{n-r}$ where $r = 0$, 1 and 2.

3 . Implementing the strategy

Given that, $q = 1 - p$

$q = 1 - 0.35$

$q = 0.65$

$P(X \leqslant 2) = P(X = 0) + P(X = 1) + P(X = 2)$

$= {}^{10}C_0(0.35)^0(0.65)^{10} + {}^{10}C_1(0.35)^1(0.65)^9 + {}^{10}C_2(0.35)^2(0.65)^8$

$= 0.0135 + 0.0725 + 0.1757$

$= 0.2617$

4 . Check and reflect

Let Y represent the number of Malaysians born between 1980 and 2000 who cannot afford to own a house.

Then, $n = 10$, $p = 0.65$ and $q = 0.35$.

$P(Y \geqslant 8) = P(Y = 8) + P(Y = 9) + P(Y = 10)$

$= {}^{10}C_8(0.65)^8(0.35)^2 + {}^{10}C_9(0.65)^9(0.35)^1 + {}^{10}C_{10}(0.65)^{10}(0.35)^0$

$= 0.1757 + 0.0725 + 0.0135$

$= 0.2617$

Self-Exercise 5.9

1. 7 students at a local university applied for state foundation scholarships. The probability that a student is awarded the scholarship is $\frac{1}{3}$. Find the probability that

(a) all of them are awarded the scholarships,

(b) only two students are awarded the scholarships,

(c) at most two students are awarded the scholarships.

2. In a game, participants have to guess the number of marbles in a bottle. The probability of guessing correctly is p.

(a) Find the value of p and the number of guesses so that the mean and the variance are 36 and 14.4 respectively.

(b) If a participant can make eight guesses, find the probability that four of them are correct.

3. 80% of pupils in a certain school are interested in science. A sample consists of n pupils are randomly selected from the school.

(a) If the probability that all the pupils selected are interested in science is 0.1342, find the value of n.

(b) Based on the answer in (a), find the probability that there are less than three pupils interested in science.

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1. A fair coin is tossed four times. Construct a probability distribution table of getting the tails.

2. A fair dice is tossed 3 times. Construct a table and draw a probability distribution graph of getting a number greater than 3.

3. The probability that a pupil continues his studies after Form 5 is 0.85. A sample of eight Form 5 pupils is chosen at random. Find the probability that
(a) all these pupils continue their studies after Form 5,
(b) less than three pupils continue their studies after Form 5.

4. A durian is randomly chosen from a few baskets. The probability that a durian chosen at random is rotten is 0.1. Find the expected value and the standard deviation of the number of rotten durians in a sample of 50 durians.



5. The binomial random variable $X \sim \text{B}(n, p)$ has a mean of 5 and a variance of 4.
(a) Find the values of n and p.
(b) Then, find $P(X = 3)$.

6. X is a discrete random variable so that $X \sim \text{B}(10, p)$ with $p < 0.5$ and variance $= \frac{12}{5}$. Find
(a) the value of p and the mean of X,
(b) $P(X = 4)$.

7. 20 pieces of fair coins are tossed simultaneously. X is a discrete random variable representing the number of tails obtained. Calculate the mean and the variance of X.

8. In a survey, it is found that 1 out of 5 brand A calculators have a life span of more than 8 years. A sample consisting of n brand A calculators is chosen at random. If the probability that all the calculators lasted more than 8 years is 0.0016, find
(a) the value of n,
(b) the probability that more than one calculator lasted more than 8 years.

9. A test consists of 16 multiple choice questions and each question has four choices, one of which is correct. A pupil guesses the answer to every question.
(a) Estimate the number of questions guessed wrongly.
(b) Find the probability that the pupil
(i) guesses wrongly in all the questions,
(ii) passes the test if 60% is the passing mark.

5.3 Normal Distribution

The properties of normal distribution graph

From the binomial distribution that you have studied, the size of the samples chosen are usually not big. Consider the following situation:

> If a sample of size n becomes large, for example $n > 30$ and $p = 0.5$, what will happen if we calculate its distribution by using binomial distribution method?

When the sample n becomes large, the calculation can become complex and the values cannot be obtained from the binomial table. So, when the sample size n becomes large, we can estimate the answer by using a normal distribution.

Flash Quiz

Give four examples of natural phenomena that can be represented by a normal distribution.

Below are the conditions needed to determine whether the size n is large enough or not.

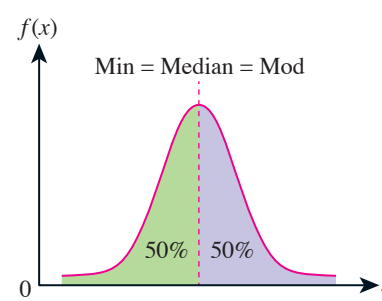
- $np \geqslant 10$, where p is the probability of 'success'.
- $n(1 - p) \geqslant 10$, where $(1 - p)$ is the probability of 'failure'.

In general,

A **normal distribution** is a probability function of a continuous random variable. The distribution is symmetrical with most of the data clustered around the centre close to the mean. The probabilities for the data further from the mean taper off equally in both directions.

The diagram on the right shows a normal distribution function graph. Based on the diagram, it shows that:

- Mean = Median = Mode
- The graph is symmetrical about an axis at the centre of the normal distribution.
- 50% of the data values is less than the mean and 50% of the data values is greater than the mean.



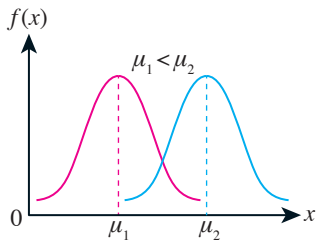
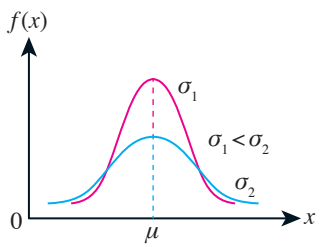
Important features of a normal distribution function graph are as follows:

- The curve is bell-shaped and is symmetrical about a vertical line that passes through the mean, μ.
- The curve has a maximum value at the axis of symmetry, $X = \mu$.
- The mean, μ divides the region under the graph into two equal parts.
- Both ends of the curve extend indefinitely without touching the x-axis.
- The total area under the graph is equal to the total probability of all outcomes, that is, 1 unit^2.

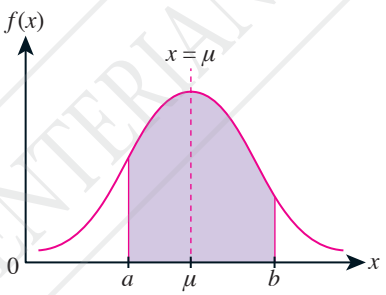
In general, the notation used for a continuous random variable X which has a normal distribution is $X \sim N(\mu, \sigma^2)$.

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Although normal distribution function graphs have similar shapes, their positions and the width of the graphs depend on their respective mean, μ and standard deviation, σ values. The table below shows the shapes and positions of normal distribution graphs when their μ and σ values change.

	The shapes and positions of normal graphs
$\mu_1 < \mu_2$ 	• The shapes of the graphs do not change. • The axis of symmetry at the mean, μ moves according to μ value when the standard deviation, σ is kept constant. • The larger the mean value, the more to the right the position of the graph.
$\sigma_1 < \sigma_2$ 	• Standard deviation affects the height and the width of a graph but the position does not change. • The larger the standard deviation value, σ, the larger the dispersion of the normal distribution from the mean value, μ. • The height of the graph increases when the standard deviation, σ value decreases if mean, μ is kept constant.

Look at the normal distribution graph below.



The area under the graph for X from a to b represents the probability of X occurring for the value of X from a to b and is written as:

$$P(a < X < b) = P(a \leqslant X \leqslant b)$$

Notice that the above two probabilities are the same since the normal distribution function is a continuous function.

Flash Quiz

What will happen to the normal distribution if $n \to \infty$?
Scan the QR code or browse the link below to explore.

ggbm.at/dkdscrnu

CHAPTER 5

KEMENTERIAN PENDIDIKAN MALAYSIA

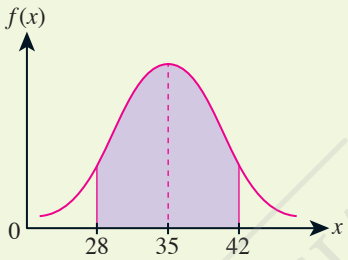
Example 14

The diagram on the right shows a normal distribution function graph which is symmetrical at $X = 35$.

(a) State the mean value, μ.

(b) Express the shaded region in probability notation.

(c) If the probability of the shaded region is 0.64, find $P(X < 28)$.



Solution

(a) $\mu = 35$

(b) $P(28 < X < 42)$

(c) Since the graph is symmetrical at $X = 35$, and $X = 28$ and $X = 42$ are both 7 units respectively to the left and right of the mean, then

$$P(X < 28) = P(X > 42)$$
$$= \frac{1 - 0.64}{2}$$
$$= 0.18$$

Information Corner

The area under the graph represents the probability of the normal distribution, that is:

$$P(-\infty < X < \infty) = 1$$

Example 15

A continuous random variable $X \sim N(2.3, 0.16)$. State the mean, μ and the standard deviation, σ for this distribution.

Solution

Given $X \sim N(2.3, 0.16)$

Then,

Mean, $\mu = 2.3$

Standard deviation, $\sigma = \sqrt{0.16}$

$\sigma = 0.4$

Excellent Tip

The notation for the variable X which is normally distributed is written as $X \sim N(\mu, \sigma^2)$.

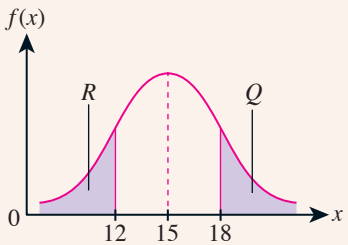
Self-Exercise 5.10

1. The diagram on the right shows a normal distribution graph for a continuous random variable X.

(a) State the mean of X.

(b) Express the shaded regions Q and R in probability notations.

(c) If $P(X < 18) = 0.7635$, find $P(X > 18)$ and $P(15 < X < 18)$.



2. A continuous random variable $X \sim N(\mu, 16)$ and is symmetrical at $X = 12$.

(a) State the value of μ.

(b) Sketch the normal distribution graph for X and shade the region representing $P(10 < X < 15)$.

Random variation and the law of large numbers

When the same experiment is repeated many times, the average result will converge to the expected result. Here, the **random variation** reduces as the number of experiments increases. This is known as the **law of large numbers**.

Consider a coin is tossed 10 times. A possible outcome obtained can be 7 times heads even though we expect only 5 heads. But, if the coin is tossed 10 000 times, the expected number will be close to 5 000 and not 7 000.

In general,

The larger the sample size, the smaller the random variation. So, the estimated value of a parameter becomes more consistent.

HISTORY GALLERY

Abraham de Moivre was a mathematician who was able to solve this problem when a sample becomes very large. He has introduced normal distribution based on the concept of the law of large numbers.

Carry out the activity below to investigate the law of large numbers.

Discovery Activity 5 Group 21st cl

Aim: To investigate the law of large numbers as the sample size grows

Steps:

1. Prepare a coin and construct a table as shown below to fill in the results for 30 flips of the coin.

Number of trials, n	Outcomes, H or T	Cumulative trial mean of getting H, μ'
1	Example: H	Obtain a head from one trial: $\frac{1}{1} = 1$
2	Example: T	Obtain a head from two trials: $\frac{1}{2} = 0.5$
3	Example: H	Obtain two heads from three trials: $\frac{2}{3} = 0.67$
⋮	⋮	⋮
30		

2. Flip the coin once. Then, record in the table whether you get a head (H) or a tail (T) like the example shown.
3. Then, calculate the mean of getting a head (H) by using the following formula.

$$\text{Mean} = \frac{\text{Number of cumulative } H \text{ obtained from } n = 1 \text{ to } n \text{ at that instant}}{\text{Number of trials at that instant, } n}$$

4. By filling up the outcome in the second column of the table, the flipping continues until $n = 30$ and calculate the mean of getting a head (H) after each flip as the example shown in the table.
5. Then, answer the following questions:
   (a) What happens to the mean value of the experiment when the number of trials increases?
   (b) It is known that the theoretical mean value, μ is 0.5. Is the experimental mean value approaching the theoretical mean value of 0.5? Explain.
   (c) From the table, draw the graph of the means of the experiment, μ' against the number of experiments, n. On the same graph, draw a straight line to represent the theoretical mean, μ, that is, 0.5.
   (d) Based on the graphs drawn, compare the experimental mean value, μ' obtained after 30 trials with the theoretical mean value, μ.
6. A representative of each group moves to other groups and presents the findings to other groups.

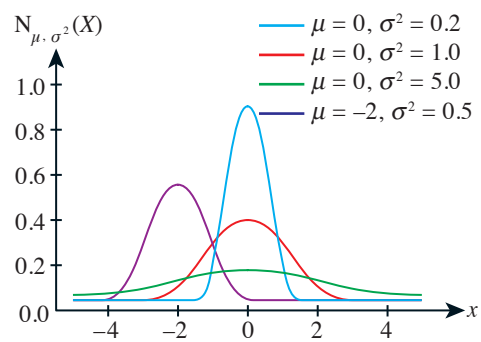
From Discovery Activity 6 results, it is found that the larger the value of n, the lower the random variation on the value of the mean. This means that the tendency of the experimental mean value to deviate from the theoretical mean reduces. The experimental mean value is said to approach the theoretical mean value.

In general,

The **law of large numbers** states that the larger the size of a sample, the value of the experimental mean gets closer to the theoretical mean value of the population.

Standard normal distribution

The diagram on the right shows four curves with normal distributions. Can all these distributions be standardised so that we can compare them?



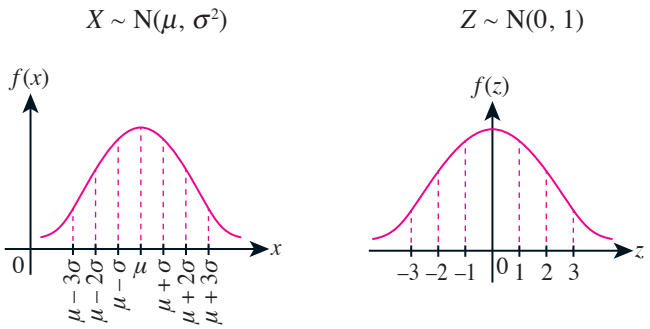
A **standard normal distribution** is defined as a normal distribution whose mean and standard deviation are 0 and 1 respectively. Based on the diagram on the right, the red curve is a standard normal distribution because its mean is 0 and it has a standard deviation of 1.

A standard normal distribution is a graph used for comparison with all other normal distribution graphs after their scores are converted to the same scale. All normal distributions can be converted to standard normal distributions with mean 0 and standard deviation 1. A continuous random variable $X \sim N(\mu, \sigma^2)$ with mean μ and standard deviation σ can be standardised by changing it to another continuous random variable Z whose mean is 0 and standard deviation is 1 by using the following formula:

$$Z = \frac{X - \mu}{\sigma}, \text{ where } Z \sim N(0, 1)$$

A continuous random variable Z is the standard normal random variable or z-score and its distribution is known as standard normal distribution.

The diagram below shows the relationship between the graphs $X \sim N(\mu, \sigma^2)$ and $Z \sim N(0, 1)$.

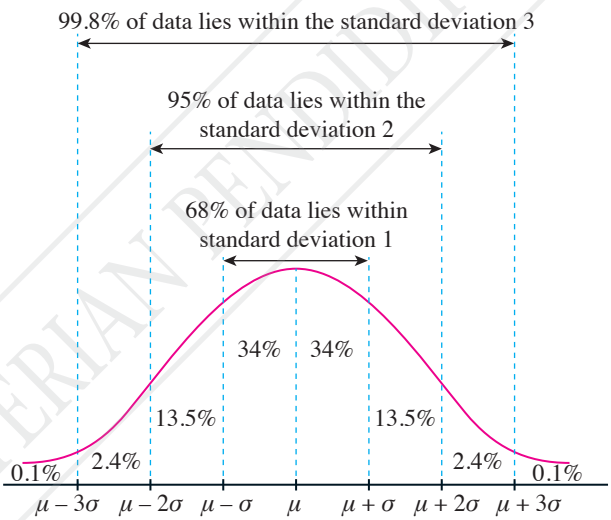


Excellent Tip

$$\text{Mean, } E(Z) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}[E(X) - \mu] = \frac{1}{\sigma}[\mu - \mu] = 0$$

$$\text{Var}(Z) = \text{Var}\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}[\text{Var}(X) - 0] = \frac{1}{\sigma^2}[\sigma^2] = 1$$

For data which are normally distributed, the standard deviation is of great importance as it measures the dispersion of the data from the mean. Typically, the percentage of data distribution within each standard deviation can be shown in the following diagram.



In general, the percentage of data distribution for each standard normal distribution is as follows:

- 68% of the data lies within the standard deviation ±1 from the mean.
- 95% of the data lies within the standard deviation ±2 from the mean.
- 99.8% of the data lies within the standard deviation ±3 from the mean.

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Determining and interpreting standard score, Z

Any continuous random variable X with a normal distribution of mean μ and a standard deviation σ can be standardised by changing to another continuous random variable Z using the formula $Z = \dfrac{X - \mu}{\sigma}$.

Example 16

(a) A continuous random variable X is normally distributed with mean 30 and a standard deviation of 8. Find the z-score if $X = 42$.

(b) The heights of buildings in Kampung Pekan are normally distributed with a mean of 23 m and a variance of 25 m^2. Find the height of the building if the standard score is 0.213.

Solution

(a) Given $X = 42$, $\mu = 30$ and $\sigma = 8$

$$Z = \frac{X - \mu}{\sigma}$$

$$Z = \frac{42 - 30}{8}$$

$$Z = 1.5$$

(b) Given $\mu = 23$, $\sigma^2 = 25$ and z-score $= 0.213$.

Then, $\sigma = \sqrt{25}$

$\sigma = 5$

Therefore,

$$Z = \frac{X - \mu}{\sigma}$$

$$0.213 = \frac{X - 23}{5}$$

$$1.065 = X - 23$$

$$X = 24.065 \text{ m}$$

Self-Exercise 5.11

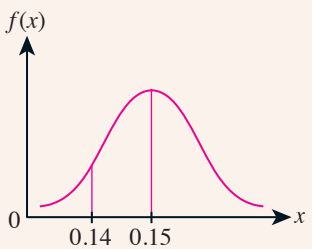
1. A continuous random variable X is normally distributed with mean, $\mu = 24$ and a standard deviation, $\sigma = 6$. Find the z-score if $X = 19.5$.

2. X is a continuous random variable that is normally distributed, such that $X \sim N(500, 169)$. Find the value of X if the z-score is 1.35.

3. The diagram on the right shows a normal distribution graph for the masses of smartphones produced by an electronic factory. If the standard deviation is 0.05 kg, find

(a) the z-score when a smartphone chosen at random has a mass of 0.14 kg,

(b) the mass of a randomly chosen smartphone if the z-score is -0.12.



4. A continuous random variable X is normally distributed and is symmetrical at $X = 45$. If X is standardised to have a standard normal distribution, it is found that $X = 60$ is standardised to $Z = 1.5$. State the mean and standard deviation of this normal distribution.

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Determining the probability of an event for normal distribution

If an event is normally distributed, then its probability can only be determined if its normal distribution is converted into standard normal distribution.

For example, to find the probability of a continuous random variable X that occurs between a and b, we write it as $P(a < X < b)$. Then, the way to convert this probability of the event to a standard normal distribution with a continuous random variable Z is as follows:

$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right)$$
$$= P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

The diagram below shows the relation between the normal distribution graph and the standard normal distribution graph.



Example 17

The lengths of a type of screw produced by a factory can be considered as normally distributed with a mean of 10.6 cm and a standard deviation of 3.2 cm. Represent the probability that a screw randomly chosen from the factory has a length between 8.4 cm and 13.2 cm where Z is a standard continuous random variable.

Solution

Let X represent the length of the screw produced by the factory.

Given $\mu = 10.6$ and $\sigma = 3.2$

P(Length of screw is between 8.4 cm and 13.2 cm) $= P(8.4 < X < 13.2)$

$$= P\left(\frac{8.4-10.6}{3.2} < \frac{X-\mu}{\sigma} < \frac{13.2-10.6}{3.2}\right)$$
$$= P(-0.6875 < Z < 0.8125)$$

The probability of z-score for a standard normal distribution, such as $P(Z > z)$ can be determined by using the standard normal distribution table. This table is formulated based on the concept that the probability of a normal distribution is the area under the curve and the total area under the graph is 1 unit^2.

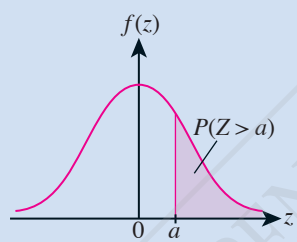
Since this graph is symmetrical, $P(Z \geqslant 0) = 0.5$ and the numeric table only gives the values of the area to the right starting with 0.5 which is for $P(Z > 0)$.

The diagram below shows a part of the standard normal distribution table.

Value of z

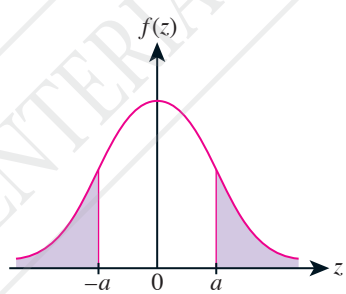
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Subtract								
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641	4	8	12	16	20	24	28	32	36
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247	4	8	12	16	20	24	28	32	36
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859	4	8	12	15	19	23	27	31	35
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483	4	7	11	15	19	22	26	30	34
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121	4	7	11	15	18	22	25	29	32
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776	3	7	10	14	17	20	24	27	31

These values give the probabilities of the standard normal distribution, that is, $P(Z > a)$.



Each of these numbers is in the value at the third or fourth decimal place. For example, 4 means 0.0004 and 19 means 0.0019.

Note that for each value of $Z = a$, it gives $P(Z > a) = P(Z < -a)$ because the standard normal distribution is symmetrical at $Z = 0$. Look at the diagram below.



Flash Quiz

If $a = 0$, what is the value for $P(Z > 0)$ or $P(Z < 0)$?

Example 18

Given that Z is a continuous random variable with a standard normal distribution, find

(a) $P(Z > 0.235)$ (b) $P(Z < -2.122)$ (c) $P(Z \geqslant -1.239)$

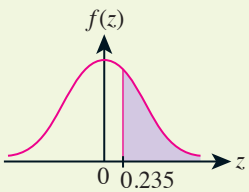
(d) $P(Z \leqslant 2.453)$ (e) $P(0 < Z < 1.236)$ (f) $P(-0.461 < Z < 1.868)$

(g) $P(|Z| > 2.063)$ (h) $P(|Z| \leqslant 1.763)$

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Solution

(a) $P(Z > 0.235)$



Flash Quiz

To find $P(Z > 0.235)$, why do we need to subtract 0.0019 from 0.4090, that is, $P(Z > 0.23)$?

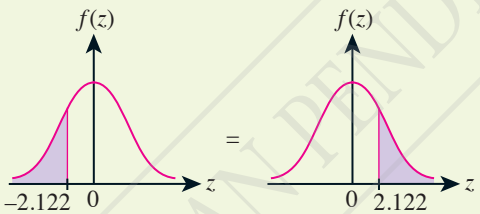
z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
														Subtract					
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641	4	8	12	16	20	24	28	32	36
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247	4	8	12	16	20	24	28	32	36
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859	4	8	12	15	19	23	27	31	35

$P(Z > 0.23) = 0.4090$

$P(Z > 0.235) = 0.4090 - 0.0019$
$= 0.4071$

Thus, $P(Z > 0.235) = 0.4071$

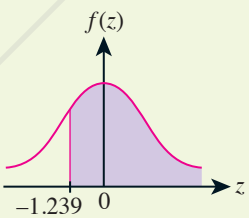
(b) $P(Z < -2.122)$
$= P(Z > 2.122)$
$= 0.0170 - 0.0001$
$= 0.0169$



Excellent Tip

Sketch a standard normal graph first before determining the probability from the standard normal distribution table.

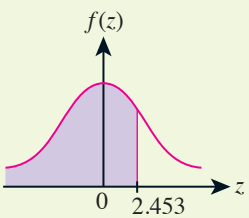
(c) $P(Z \geq -1.239)$
$= 1 - P(Z < -1.239)$
$= 1 - P(Z > 1.239)$
$= 1 - (0.1093 - 0.0017)$
$= 0.8924$



Excellent Tip

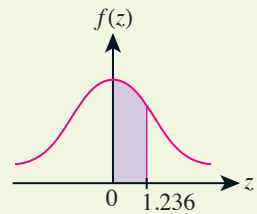
The standard normal distribution table only gives the values of the area to the right tail of the graph.

(d) $P(Z \leq 2.453)$
$= 1 - P(Z > 2.453)$
$= 1 - (0.00714 - 0.0006)$
$= 0.9935$

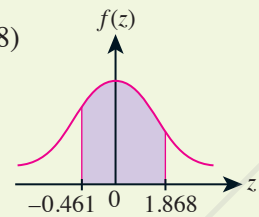


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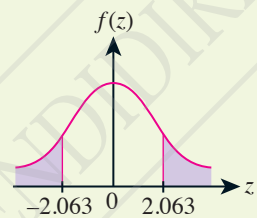
(e) $P(0 < Z < 1.236)$
$= P(Z > 0) - P(Z > 1.236)$
$= 0.5 - (0.1093 - 0.0011)$
$= 0.3918$



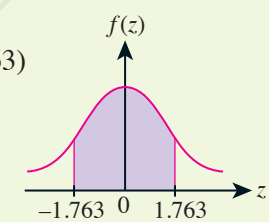
(f) $P(-0.461 < Z < 1.868)$
$= 1 - P(Z < -0.461) - P(Z > 1.868)$
$= 1 - P(Z > 0.461) - P(Z > 1.868)$
$= 1 - 0.3224 - 0.0308$
$= 0.6468$



(g) $P(|Z| > 2.063)$
$= P(Z < -2.063) + P(Z > 2.063)$
$= 2P(Z > 2.063)$
$= 2(0.0196)$
$= 0.0392$



(h) $P(|Z| \leq 1.763)$
$= P(-1.763 \leq Z \leq 1.763)$
$= 1 - P(Z < -1.763) - P(Z > 1.763)$
$= 1 - 2P(Z > 1.763)$
$= 1 - 2(0.0389)$
$= 0.9222$



Calculator Literate

To determine the solution for Example 16(e) by using a scientific calculator.

1. Press MENU 7 2 for the cumulative normal distribution.
2. Press 0 for Lower and press =
3. Press 1 . 2 3 6 for Upper and press =
4. Press = again.
5. The screen will display
P = 0.3917707182

Example 19

Find the z-score for each of the following probabilities from the standard normal distribution.

(a) $P(Z > a) = 0.3851$
(b) $P(Z < a) = 0.3851$
(c) $P(Z > a) = 0.7851$
(d) $P(-0.1 < Z \leq a) = 0.3851$
(e) $P(a < Z \leq 2.1) = 0.8633$
(f) $P(|Z| \leq a) = 0.4742$

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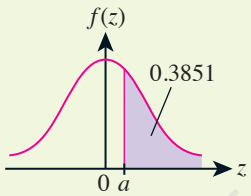
Solution

(a) $P(Z > a) = 0.3851$

$= 0.3859 - 0.0008$

From the standard normal distribution table, we get

$0.3851 = 0.3859 - 0.0008$



z	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
											Subtract								
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641	4	8	12	16	20	24	28	32	36
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247	4	8	12	16	20	24	28	32	36
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859	4	8	12	15	19	23	27	31	35

So, $a = 0.2 + 0.09 + 0.002$

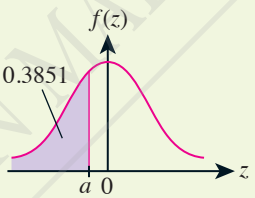
$a = 0.292$

(b) $P(Z < a) = 0.3851$

Based on the diagram on the right, a is negative.

$P(Z > a) = 0.3851$

$a = -0.292$



(c) $P(Z > a) = 0.7851$

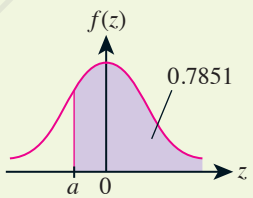
Based on the diagram on the right, a is negative because the area is more than 0.5 unit^2.

$1 - P(Z < a) = 0.7851$

$P(Z < a) = 1 - 0.7851$

$= 0.2149$

$a = -0.789$



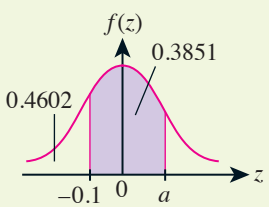
(d) $P(-0.1 < Z \leq a) = 0.3851$

$1 - P(Z < -0.1) - P(Z > a) = 0.3851$

$1 - 0.4602 - P(Z > a) = 0.3851$

$P(Z > a) = 0.1547$

$a = 1.017$



(e) $P(a < Z \leq 2.1) = 0.8633$

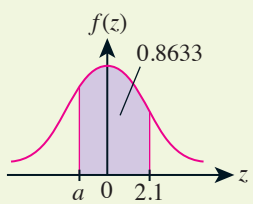
Based on the diagram on the right, a is negative because the area is more than 0.5 unit^2.

$1 - P(Z < a) - P(Z > 2.1) = 0.8633$

$1 - P(Z < a) - 0.0179 = 0.8633$

$P(Z < a) = 0.1188$

$a = -1.181$



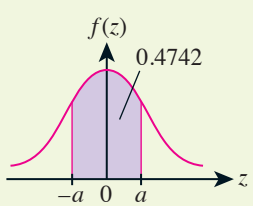
(f) $P(|Z| \leq a) = 0.4742$

Since the graph is symmetrical,

$P(Z > a) = 0.5 - \frac{1}{2}(0.4742)$

$= 0.2629$

$a = 0.634$



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Example 20

If $X \sim N(45, \sigma^2)$ and $P(X > 51) = 0.2888$, find the value of σ.

Solution

Given $\mu = 45$

$$P(X > 51) = 0.2888$$

Standardise X to Z,

$$P\left(\frac{X-\mu}{\sigma} > \frac{51-45}{\sigma}\right) = 0.2888$$

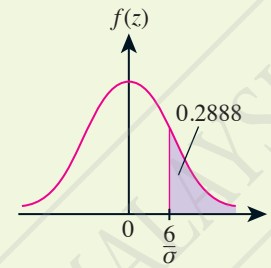
$$P\left(Z > \frac{6}{\sigma}\right) = 0.2888$$

$$\frac{6}{\sigma} = 0.557$$

$$\sigma = \frac{6}{0.557}$$

$$\sigma = 10.77$$

0.557 is the z-score obtained from the standard normal distribution table



Example 21

A continuous random variable X is normally distributed with a mean μ and a variance of 12. Given that $P(X > 32) = 0.8438$, find the value of μ.

Solution

Given $\sigma^2 = 12$

$$P(X > 32) = 0.8438$$

Standardise X to Z,

$$P\left(\frac{X-\mu}{\sigma} > \frac{32-\mu}{\sqrt{12}}\right) = 0.8438$$

$$P\left(Z > \frac{32-\mu}{\sqrt{12}}\right) = 0.8438$$

$$1 - P\left(Z < -\frac{32-\mu}{\sqrt{12}}\right) = 0.8438$$

$$P\left(Z < -\frac{32-\mu}{\sqrt{12}}\right) = 1 - 0.8438$$

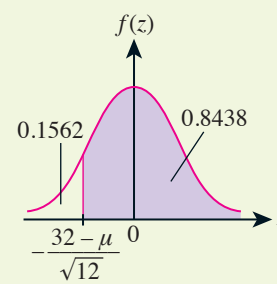
$$P\left(Z < -\frac{32-\mu}{\sqrt{12}}\right) = 0.1562$$

$$-\frac{32-\mu}{\sqrt{12}} = 1.01$$

$$\mu = 32 + 1.01\left(\sqrt{12}\right)$$

$$\mu = 35.50$$

1.01 is the z-score obtained from the standard normal distribution table



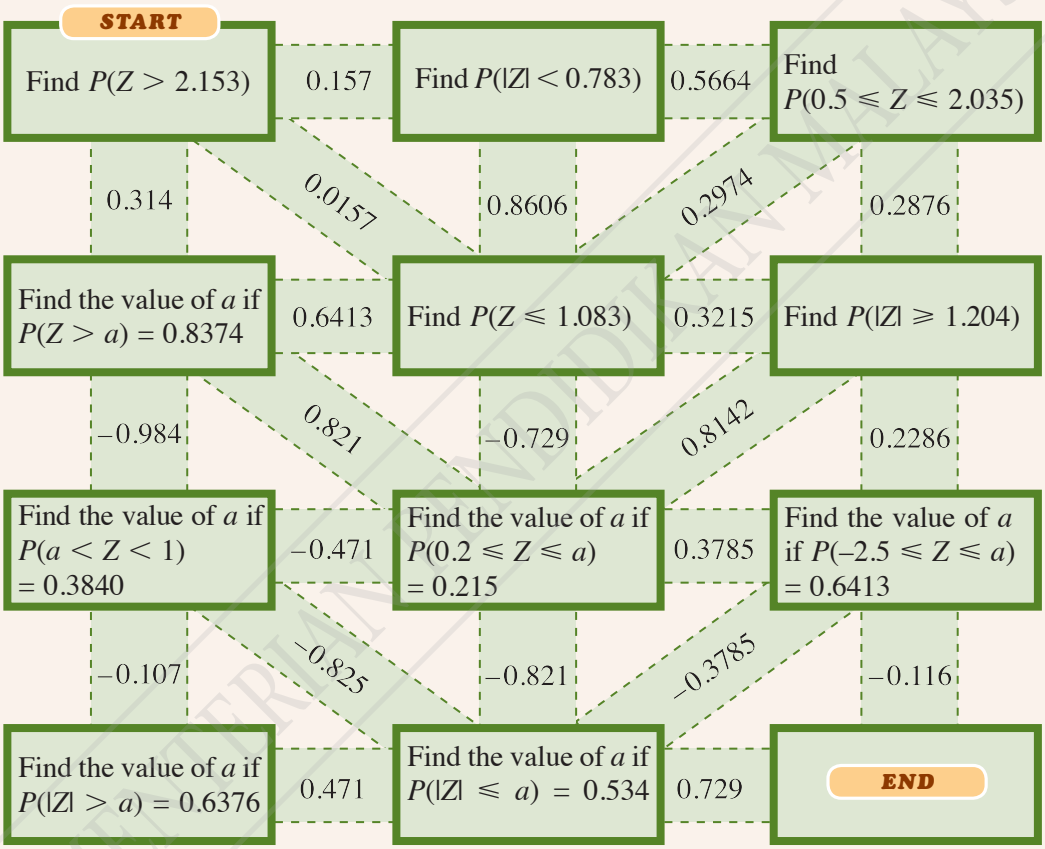
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Self-Exercise 5.12

1. The masses of bread baked by company M are normally distributed with a mean of 350 g and a standard deviation of 45 g. Convert the probability of a loaf of bread randomly selected from company M that has a mass between 280 g and 375 g where Z is a standard continuous random variable.

2. Given Z is a continuous random variable for the standard normal distribution, find
(a) $P(Z \leq 0.538)$ (b) $P(-2.1 < Z < 1.2)$
(c) $P(-1.52 < Z < -0.253)$ (d) $P(0 \leq Z \leq 1.984)$

3. Find the path to the END of the maze by choosing the correct answers.



4. Z is a continuous random variable for a standard normal distribution. Find the value of k when
(a) $P(Z < k) = 0.6078$ (b) $P(Z \geq k) = 0.4538$

5. If a continuous random variable X has a normal distribution with a mean of 15 and a variance of σ^2 and $P(X < 16.2) = 0.7654$, find the value of σ.

6. A continuous random variable X is normally distributed with a mean of 0.75 and a standard deviation of σ. Given $P(X > 0.69) = 0.5178$, find the value of σ.

7. If $Y \sim N(\mu, 16)$ and $P(Y > 14.5) = 0.7321$, find the value of μ.

8. Given $X \sim N(\mu, \sigma^2)$ with $P(X > 80) = 0.0113$ and $P(X < 30) = 0.0287$, find the value of μ and σ.

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Solving problems involving normal distributions

Example 22

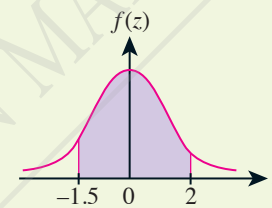
The thickness of papers produced by a machine is normally distributed with a mean of 1.05 mm and a standard deviation of 0.02 mm. Determine the probability that a piece of paper chosen randomly will have a thickness
(a) between 1.02 mm and 1.09 mm,
(b) more than 1.08 mm or less than 0.992 mm.

Solution

Given $\mu = 1.05$ mm and $\sigma = 0.02$ mm for a normal distribution.
Let X be a continuous random variable that represents the thickness of the paper.

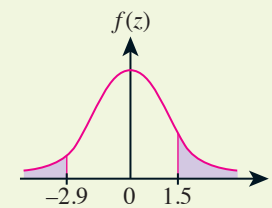
(a) $P(1.02 < X < 1.09)$

$$= P\left(\frac{1.02 - 1.05}{0.02} < \frac{X - \mu}{\sigma} < \frac{1.09 - 1.05}{0.02}\right)$$
$$= P(-1.5 < Z < 2)$$
$$= 1 - P(Z > 2) - P(Z > 1.5)$$
$$= 1 - 0.0228 - 0.0668$$
$$= 0.9104$$



(b) $P(X > 1.08)$ or $P(X < 0.992)$

$$= P\left(\frac{X - \mu}{\sigma} > \frac{1.08 - 1.05}{0.02}\right) + P\left(\frac{X - \mu}{\sigma} < \frac{0.992 - 1.05}{0.02}\right)$$
$$= P(Z > 1.5) + P(Z < -2.9)$$
$$= P(Z > 1.5) + P(Z > 2.9)$$
$$= 0.0668 + 0.00187$$
$$= 0.0687$$



Example 23 MATHEMATICAL APPLICATIONS

The masses of chickens reared by Mr Rahmat are normally distributed with a mean of 1.2 kg and a standard deviation of 0.3 kg.
(a) If Mr Rahmat rears 1 500 chickens, find the number of chickens whose masses are between 0.95 kg and 1.18 kg.
(b) Given that 10% of the chicken have masses less than m kg, find the value of m.



Solution

1 . Understanding the problem

Given $\mu = 1.2$ kg and $\sigma = 0.3$ kg for a normal distribution.
Let X represent the masses of chickens reared by Mr Rahmat.
(a) If the number of chickens raised is 1 500, find the number of chickens with $P(0.95 < X < 1.18)$.
(b) Find the value of m for $P(X < m) = 0.1$.

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2 . Planning the strategy

- Convert the variable X to z-score.
- Sketch a normal distribution graph to determine the region required.
- Use the standard normal distribution table or a calculator to find the probability.

3 . Implementing the strategy

(a) $P(0.95 < X < 1.18)$

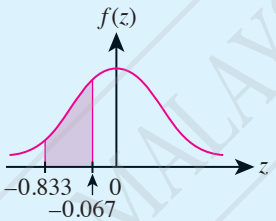
$= P\left(\frac{0.95 - 1.2}{0.3} < Z < \frac{1.18 - 1.2}{0.3}\right)$

$= P(-0.833 < Z < -0.067)$

$= P(Z > 0.067) - P(Z > 0.833)$

$= 0.4733 - 0.2025$

$= 0.2708$



So, the number of chickens with masses between 0.95 kg and 1.18 kg

$= 0.2708 \times 1\,500$

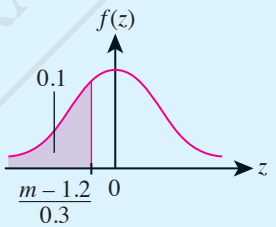
$= 406$

(b) $P(X < m) = 0.1$

$P\left(Z < \frac{m - 1.2}{0.3}\right) = 0.1$

$\frac{m - 1.2}{0.3} = -1.281$

$m = 0.8157$



4 . Check and reflect

(a) If there are 406 chickens with masses between 0.95 kg and b kg, then

$P(0.95 < X < b) \times 1\,500 = 406$

$P(0.95 < X < b) = 0.2707$

$P\left(\frac{0.95 - 1.2}{0.3} < Z < \frac{b - 1.2}{0.3}\right) = 0.2707$

$P\left(-0.833 < Z < \frac{b - 1.2}{0.3}\right) = 0.2707$

$P\left(Z > \frac{b - 1.2}{0.3}\right) - P(Z > 0.833) = 0.2707$

$P\left(Z > \frac{b - 1.2}{0.3}\right) - 0.2025 = 0.2707$

$P\left(Z > \frac{b - 1.2}{0.3}\right) = 0.4732$

$\frac{b - 1.2}{0.3} = -0.067$

$b = 1.18$ kg

(b) $P(X < 0.8157)$

$= P\left(Z < \frac{0.8157 - 1.2}{0.3}\right)$

$= P(Z < -1.281)$

$= P(Z > 1.281)$

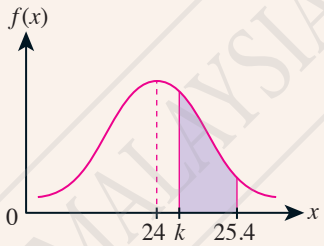
$= 0.1$

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Self-Exercise 5.13

1. Given X is a continuous random variable that is normally distributed with a mean of 210 and a standard deviation of 12, find
(a) the z-score if $X = 216$,
(b) X if the z-score is -1.8.

2. The diameters of basketballs produced by a factory are normally distributed with a mean of 24 cm and a standard deviation of 0.5 cm. The diagram on the right shows the normal distribution graph for the diameters, in cm, of the basketballs. Given that the area of the shaded region is 0.245, find the value of k.



3. The heights of Form 1 pupils in a certain school are normally distributed with a mean of 145 cm and a standard deviation of 10 cm.
(a) If a pupil is randomly selected from that group, find the probability that the pupil's height is at least 140 cm.
(b) If there are 450 pupils in Form 1, find the number of pupils with the height not more than 150 cm.

4. In a certain school, 200 pupils took a mathematics test. The scores are normally distributed with a mean of 50 marks and a standard deviation of 10 marks.
(a) In the test, pupils who obtained 70 marks and above are categorised as excellent. Find the number of pupils in that category.
(b) Given that 60% of pupils passed the test, calculate the minimum score to pass.

5. The marks in an English test in a school are normally distributed with a mean μ and a variance σ^2. 10% of the pupils in that school scored more than 75 marks and 25% of the pupils scored less than 40 marks. Find the values of μ and σ.

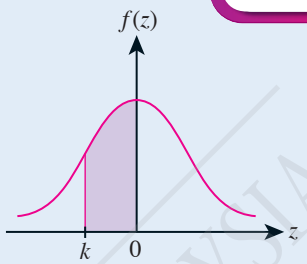
6. The masses of papayas produced in an orchard have a normal distribution with a mean of 840 g and a standard deviation of 24 g. The papayas with masses between 812 g and 882 g will be exported overseas while papayas that weigh 812 g or less will be sold at the local market. Find
(a) the probability that a papaya chosen at random to be exported overseas,
(b) the number of papayas which are not exported overseas and not sold in the local market if the orchard produces 2 500 papayas.

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Formative Exercise 5.3

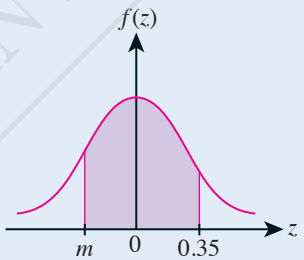
Quiz bit.ly/31nGeFJ

1. The diagram on the right shows a standard normal distribution graph. The probability represented by the shaded region is 0.3415. Find the value of k.



2. X is a continuous random variable that is normally distributed with a mean of 12 and a variance of 4. Find
(a) the z-score if $X = 14.2$,
(b) $P(11 < X < 13.5)$.

3. The diagram on the right shows a standard normal distribution graph. If $P(m < Z < 0.35) = 0.5124$, find $P(Z < m)$.



4. The masses of babies born in a hospital are normally distributed with a mean of 3.1 kg and a standard deviation of 0.3 kg.
(a) Find the probability that a baby born in that hospital has a mass between 2.9 kg and 3.3 kg.
(b) If 25% of babies born in that hospital are categorised as underweight, find the maximum mass for this category.

5. The photo on the right shows the fish reared by Mr Lim. The masses of fish in the pond are normally distributed with a mean of 650 g and a standard deviation of p g.
(a) If the probability that a fish caught randomly has a mass of less than 600 g is 0.0012, find the value of p.
(b) If 350 fish have masses between 645 g and 660 g, find the number of fish in the pond.



6. The daily wages of workers in a factory are normally distributed with a mean of RM80 and a standard deviation of RM15.
(a) Given that the number of workers in the factory is 200, find the number of workers whose daily wages are more than RM85.
(b) Find the value of p if p% of the workers in the factory earn less than RM85.

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REFLECTION CORNER

PROBABILITY DISTRIBUTION

Discrete random variable

$$\sum_{i=1}^{n} P(X = r_i) = 1$$

The probability distribution can be interpreted by a tree diagram, a table or a graph.

Binomial distribution, $X \sim B(n, p)$

- Involves n Bernoulli trials which are similar.
- $P(X = r) = {}^nC_r p^r q^{n-r}$ where
  n = number of trials
  r = number of 'success'
  $= 0, 1, 2, \ldots, n$
  p = probability of 'success'
  q = probability of 'failure'
  $= 1 - p$

Mean, variance and standard deviation

- Mean, $\mu = np$
- Variance, $\sigma^2 = npq$
- Standard deviation, $\sigma = \sqrt{npq}$

Applications

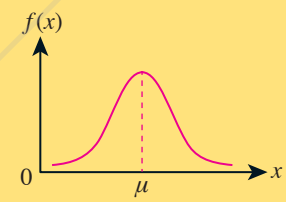
$n > 30$

Continuous random variable

$$P(-\infty < X < \infty) = 1$$

The probability distribution can be interpreted by using a continuous graph.

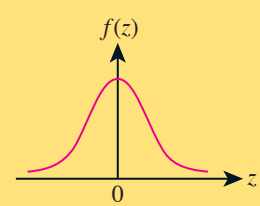
Normal distribution, $X \sim N(\mu, \sigma^2)$



- Bell-shaped
- Symmetrical at $X = \mu$ axis.
- Area under the graph for $-\infty < X < \infty$ represents the probability which is given by $P(-\infty < X < \infty) = 1$

Standard normal distribution, $Z \sim N(0, 1)$

A standard continuous random variable, $Z = \dfrac{X - \mu}{\sigma}$.



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Journal Writing

Construct a graphic info on the characteristics, types of probability distributions and the relation between discrete random variables and continuous random variables. Next, find information from the Internet on the importance of normal distribution in daily lives.

Summative Exercise

1. Two fair dice are tossed at the same time. Number A and number B on the surface on both dice are recorded. Let X represent the scores which are defined by $X = \{A + B: A = B\}$. List all the possible values of X. PL 1

2. The table below shows the probability distribution of a discrete random variable X. PL 2

$X = r$	1	2	3	4
$P(X = r)$	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{1}{3}$	q

(a) Find the value of q.
(b) Find $P(X > 2)$.

3. A school implements a merit and demerit system. In that system, each pupil will be given 2 points if he behaves well and –1 points if he behaves badly for each week. Let '+' represent good behaviour and '–' represent bad behaviour. PL 3
(a) Construct a tree diagram to show all the possible behaviours of a pupil randomly selected from the school for a period of 3 weeks.
(b) If X represents the points a pupil receives during the 3 weeks, list all the possible outcomes for X in a set notation.

4. In a game, a player is required to throw tennis balls into a basket from a certain distance. Each player is given 3 attempts. The probability that a player succeeds in throwing a tennis ball into the basket is 0.45. PL 3
(a) If X represents the number of times a tennis ball enters the basket, show that X is a discrete random variable.
(b) List all the possible outcomes in one table and then draw a graph to represent the probabilities.

5. If $X \sim B(6, 0.4)$, find PL 2
(a) $P(X = 2)$
(b) $P(X > 4)$
(c) $P(X \leq 2)$

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6. The probability that a housewife buys the *W* brand detergent is 0.6. A sample of 8 housewives were randomly selected. Find the probability that PL 3

(a) exactly 3 housewives buy the *W* brand detergent,

(b) more than 4 housewives buy the *W* brand detergent.



7. In a survey, it is found that 18 out of 30 college students have reading as their hobby. If 9 students are selected at random, find the probability that PL 3

(a) exactly 4 students have reading as their hobby,

(b) at least 7 students have reading as their hobby.

8. A farmer picks mangosteens at random from an orchard. The probability that a mangosteen has worms is $\frac{1}{5}$. Find the mean and standard deviation of the number of mangosteens with worms in a sample of 35 mangosteens. PL 2

9. In a group of teachers, the mean number of teachers who own local cars is 7 and the variance is 2.8. Find the probability that PL 3

(a) a randomly selected teacher owns a local car,

(b) 2 randomly selected teachers own local cars.

10. Given $X \sim \text{N}(48, 144)$, find the value of k if PL 3

(a) $P(X > 47) = k$

(b) $P(38 < X < 46) = k$

(c) $P(X \leqslant 49.5) = k$

(d) $P(47 < X < 50) = k$

(e) $P(X > k) = 0.615$

(f) $P(45 < X < k) = 0.428$

(g) $P(X > |k|) = 0.435$

(h) $P(-k < X < 48) = 0.2578$

11. It is known that the intelligence quotient (IQ) test results of 500 candidates who applied to enter a teachers' training college are normally distributed with a mean of 115 and a standard deviation of 10. PL 4

(a) If the college requires an IQ of not less than 96, estimate the number of candidates who do not qualify to enter the college.

(b) If 300 candidates are qualified to enter the college, find the minimum IQ value needed.

12. A body mass check is performed on workers in a factory. The body masses of workers in the factory are normally distributed with a mean of 65 kg and a variance of 56.25 kg^2. There are 250 workers with body masses between 56 kg and 72 kg. PL 5

(a) Find the number of workers in the factory.

(b) If 5% of workers are obese, find the minimum body mass for this category.

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13. An orchard produces oranges. The table below shows the grading of the oranges to be marketed according to their masses. PL 5

Grade	A	B	C
Mass, X (g)	$X > 300$	$200 < X \leqslant 300$	$m < X \leqslant 200$

It is given that the masses of oranges produced in the orchard are normally distributed with a mean of 260 g and a standard deviation of 35 g.

(a) If an orange is chosen at random, find the probability that it is from the grade A.

(b) A basket has 600 oranges, estimate the number of grade B oranges.

(c) If 99% of the oranges can be graded and sold, find the minimum possible mass that can be graded and sold.

MATHEMATICAL EXPLORATION

How do you know how many candies are in a bottle without having to count them one by one? Let's do the following activity in groups.

1. Prepare a bottle of candies of various colours without the blue coloured ones and 30 blue candies.

2. Follow the steps below.

- Remove 30 random candies from the bottle and replace them with the 30 blue candies.
- Shake the bottle so that the blue candies are mixed uniformly in the bottle.
- Remove one spoonful of candies from the bottle as a random sample.
- Count the number of candies, n which have been taken out and also the number of blue candies, m among them. Then, find the ratio of $\frac{m}{n}$.
- Put the candies back into the bottle and shake it well.



3. Repeat the steps above for the second random sample until the 10th random sample so as to reduce the random variation on the value of $\frac{m}{n}$.

4. Then, estimate the number of candies in the bottle by using the method from Discovery Activity 6.

5. Check your answer by dividing the candies into several portions and ask friends from other groups to count them.

6. Using the concept derived from the activities above, help each of the following companies to solve the problems they are facing.

(a) How can a car manufacturer know what car colour Malaysians like?

(b) How can a smartphone importer company know which smartphone brand the majority of users prefer?

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CHAPTER 6

TRIGONOMETRIC FUNCTIONS



What will be learnt?

- Positive Angles and Negative Angles
- Trigonometric Ratios of Any Angle
- Graphs of Sinus, Cosine and Tangent Functions
- Basic Identities
- Addition Formulae and Double Angle Formulae
- Application of Trigonometric Functions

List of Learning Standards

bit.ly/32RJbxR

Kuala Terengganu Drawbridge crosses Sungai Terengganu's estuary and links Kuala Nerus with Kuala Terengganu. The 638-metre-long and 23-metre-wide bridge uses Bascule Bridge or Drawbridge concept. The trigonometric concept involving angles is used to calculate the torques and the forces involved in the construction of the bridge. What information is needed to calculate the width of the passage for ships when the bridge is in use? What are the common trigonometric formulae used?

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Info Corner

Abu Abdullah Muhammad Ibn Jabir Ibn Sinan al-Battani al-Harrani (858-929 C) was a mathematician who was an expert in the field of trigonometry.

He established trigonometry to a higher level and was the first to produce the cotangent table.

For more info:

bit.ly/3ksvSLd

Significance of the Chapter

The concept of trigonometry is useful in solving daily life problems. For example:

- The field of astronomy uses the concept of triangles to determine the position of places on the latitudes and longitudes
- The field of cartography to draw maps
- Oceanography field to determine sea waves height
- Military and aviation fields

Key words

Degree	*Darjah*
Radian	*Radian*
Trigonometric ratio	*Nisbah trigonometri*
Quadrant	*Sukuan*
Basic identities	*Identiti asas*
Complementary angle formula	*Rumus sudut pelengkap*
Addition angle formula	*Rumus sudut majmuk*
Double angle formula	*Rumus sudut berganda*

Video about Terengganu Drawbridge

bit.ly/398i9Vk

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6.1 Positive Angles and Negative Angles

Representing the positive and negative angles in a Cartesian plane

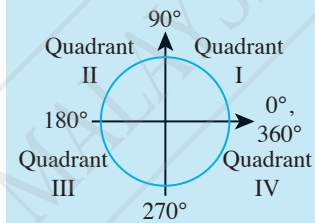
In daily life, there are many things that rotate either in the clockwise or anticlockwise direction. The minute and the hour hands of a clock move in a clockwise direction. Look at the clock in the diagram below.



What directions are represented by the red and the blue arrows? The blue arrow is the clockwise direction while the red arrow is the anticlockwise direction.

Recall

Location of angles can be specified in terms of quadrants.



In trigonometry,

- **Positive angles** are angles measured in the anticlockwise direction from the positive x-axis.
- **Negative angles** are angles measured in the clockwise direction from the positive x-axis.

Flash Quiz

Given π rad = 180°. Convert each of the following angles into radians.

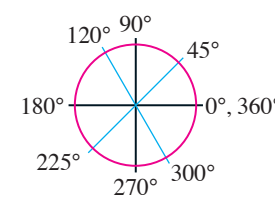


Diagram 6.1 and Diagram 6.2 show positive and negative angles formed in a quadrant, a semicircle, three quarter of a circle and a full circle when the OP line rotates in the anticlockwise and clockwise directions from the positive x-axis respectively.

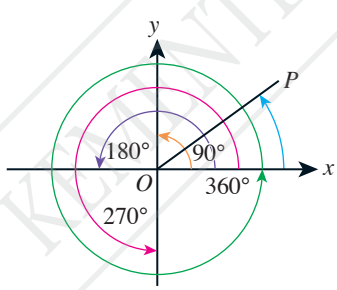


Diagram 6.1

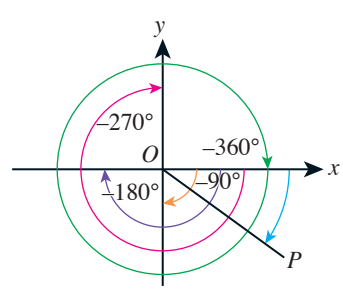


Diagram 6.2

You have learnt that a full circle contains 360° and angles can be measured in degrees, minutes and radians. What is the relation between the angles measured in degrees, in minutes and in radians? How do we determine the positions of angles in the quadrants?

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Discovery Activity 1

Group | 21st cl | STEM | CT

Aim: To explore positive and negative angles and to determine their positions in the quadrants

ggbm.at/rgyw7baz

Steps:

1. Scan the QR code or visit the link next to it.
2. Click the positive orientation button and drag the slider to the left and right.
3. Click also the negative orientation button and drag the slider to the left and right.
4. Identify the difference between the angle in the positive orientation and the angle in the negative orientation.
5. Copy and complete the table below by determining the positions of each angles.

Angle	Quadrant	Angle	Quadrant	Angle	Quadrant
140°		1 000°		−550°	
$\frac{7}{6}\pi$ rad		$\frac{13}{2}\pi$ rad		$-\frac{16}{3}\pi$ rad	
500°		−135°		−850°	
$\frac{11}{6}\pi$ rad		$-\frac{5}{6}\pi$ rad		$-\frac{27}{8}\pi$ rad	

6. Compare your group's results with other groups.
7. Then, present the result to the class.

From the result of Discovery Activity 1, it is found that an angle whether it is positive or negative can lie in any of the four quadrants. A complete cycle occurs when a line rotates through 360° or 2π rad about the origin O. If the line rotates more than one cycle, the angle formed is greater than 360° or 2π rad.

The position of an angle can be shown on a Cartesian plane.

In general,

If θ is an angle in a quadrant such that $> 360°$, then the position of θ can be determined by subtracting a multiple of 360° or 2π rad to obtain an angle that corresponds to $0° \leqslant \theta \leqslant 360°$ or $0 \leqslant \theta \leqslant 2\pi$ rad.

Excellent Tip

The position of an angle can be specified by turning the angle in radian unit to degree unit.

$60' = 1°$

$\theta° = \left(\theta° \times \frac{\pi}{180°}\right)$ rad

θ rad $= \left(\theta \text{ rad} \times \frac{180}{\pi}\right)°$

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Example 1

Determine the position of each of the following angles in the quadrants. Then, show that angle on a Cartesian plane.

(a) 800°

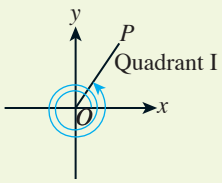
(b) $\frac{19}{6}\pi$ rad

Solution

(a) $800° - 2(360°) = 80°$

$800° = 2(360°) + 80°$

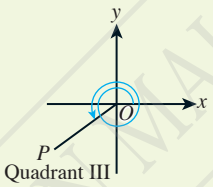
Thus, 800 lies in Quadrant I.



(b) $\frac{19}{6}\pi$ rad $- 2\pi$ rad $= \frac{7}{6}\pi$ rad

$\frac{19}{6}\pi$ rad $= 2\pi$ rad $+ \frac{7}{6}\pi$ rad

Thus, $\frac{19}{6}\pi$ rad lies in Quadrant III.



Self-Exercise 6.1

1. Convert the following angles to radians.

(a) 290° 10′ (b) −359.4° (c) 620° (d) −790°

2. Convert the following angles to degrees.

(a) 1.3 rad (b) $\frac{13}{4}$ rad (c) -2.7π rad (d) $\frac{13}{4}\pi$ rad

3. Determine the quadrant for each of the following angles. Hence, represent each angle on a separate Cartesian plane.

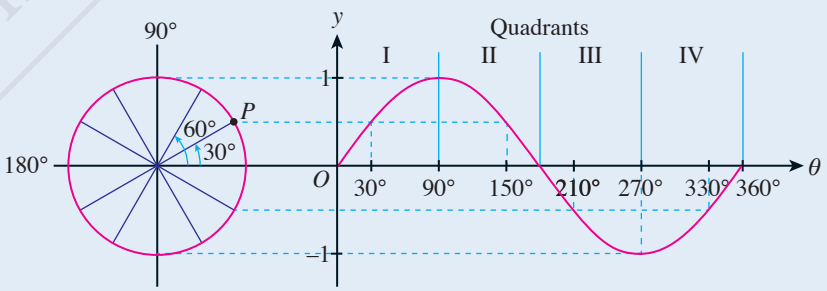
(a) 75° (b) −340.5° (c) 550° (d) −735°

(e) 0.36 rad (f) −4 rad (g) $\frac{5}{3}\pi$ rad (h) $-\frac{20}{3}\pi$ rad

Formative Exercise 6.1

Quiz bit.ly/36V31vC

1. The diagram below shows the graph $y = \sin\theta$ for $0° \leqslant \theta \leqslant 360°$.



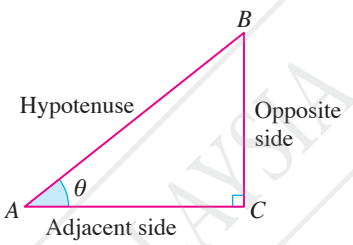
Convert each angle on the θ-axis to radians. Then, show each angle on a separate Cartesian plane.

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6.2 Trigonometric Ratio of Any Angle

Relate secant, cosecant and cotangent with sine, cosine and tangent of any angle in a Cartesian plane

Consider the triangle ABC in the diagram on the right.
The trigonometric ratios can be defined as follows:



- $\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{BC}{AB}$
- $\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{AC}{AB}$
- $\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{BC}{AC}$

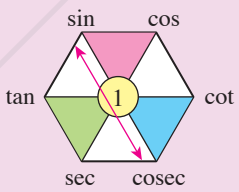
Besides the three trigonometric ratios above, there are three more ratios that are the reciprocals of these trigonometric ratios. These trigonometric ratios are cosecant, secant and cotangent which are defined as follows:

- $\text{cosec}\ \theta = \dfrac{\text{hypotenuse}}{\text{opposite side}} = \dfrac{AB}{BC}$
- $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent side}} = \dfrac{AB}{AC}$
- $\cot\theta = \dfrac{\text{adjacent side}}{\text{opposite side}} = \dfrac{AC}{BC}$

Based on the triangle ABC, it is found that:

$\text{cosec}\ \theta = \dfrac{1}{\sin\theta}$ $\quad$ $\sec\theta = \dfrac{1}{\cos\theta}$ $\quad$ $\cot\theta = \dfrac{1}{\tan\theta}$

Excellent Tip



Given A is an angle, then

$\sin A = \dfrac{1}{\text{cosec}\ A}$

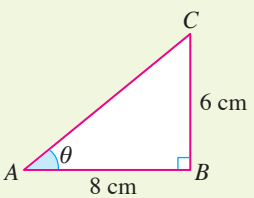
$\text{cosec}\ A = \dfrac{1}{\sin A}$

$\cot A = \dfrac{1}{\tan A}$

CHAPTER 6

Example 2

The diagram on the right shows a right-angled triangle ABC at B.
Given $AB = 8$ cm and $BC = 6$ cm, determine the value of
(a) $\text{cosec}\ \theta$ (b) $\sec\theta$ (c) $\cot\theta$



Solution

By using Pythagoras's theorem, $AC = \sqrt{6^2 + 8^2}$
$= 10$ cm

(a) $\text{cosec}\ \theta = \dfrac{10}{6}$
$= 1.667$

(b) $\sec\theta = \dfrac{10}{8}$
$= 1.25$

(c) $\cot\theta = \dfrac{8}{6}$
$= 1.333$

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Example 3

Given $\alpha = 56°$. Use a calculator to find the value of

(a) cosec α (b) sec α (c) cot α

Solution

(a) $\text{cosec } 56° = \dfrac{1}{\sin 56°}$
$= 1.206$

(b) $\sec 56° = \dfrac{1}{\cos 56°}$
$= 1.788$

(c) $\cot 56° = \dfrac{1}{\tan 56°}$
$= 0.675$

The angles A and B are **complementary angles** to each other if $A + B = 90°$.
Hence,

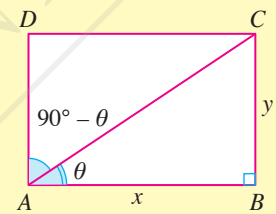
$A = 90° - B$ and $B = 90° - A$

Discovery Activity 2

Group 21st cl

Aim: To formulate the complementary angle formulae

Steps:



1. Consider the rectangle $ABCD$ in the diagram on the right. Then, complete all the lengths of the sides of the rectangle $ABCD$.
2. Copy and complete the table below in terms of x and y.

Column A	Column B
$\sin \theta =$	$\sin (90° - \theta) =$
$\cos \theta =$	$\cos (90° - \theta) =$
$\tan \theta =$	$\tan (90° - \theta) =$
$\cot \theta =$	$\cot (90° - \theta) =$
$\sec \theta =$	$\sec (90° - \theta) =$
$\text{cosec } \theta =$	$\text{cosec } (90° - \theta) =$

3. Based on the table above, map the trigonometric ratios in column A to the trigonometric ratios in column B.
4. Then compare your results with other groups and draw conclusions from the comparisons.

From the results of Discovery Activity 2, the formulae of the complementary angles are as follows:

- $\sin \theta = \cos (90° - \theta)$
- $\cos \theta = \sin (90° - \theta)$
- $\tan \theta = \cot (90° - \theta)$
- $\sec \theta = \text{cosec } (90° - \theta)$
- $\text{cosec } \theta = \sec (90° - \theta)$
- $\cot \theta = \tan (90° - \theta)$

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Example 4

Given that $\sin 77° = 0.9744$ and $\cos 77° = 0.225$. Find the value of each of the following.
(a) $\cos 13°$ (b) $\operatorname{cosec} 13°$ (c) $\cot 13°$

Solution

(a) $\cos 13° = \sin (90° - 13°)$
$= \sin 77°$
$= 0.9744$

(b) $\operatorname{cosec} 13° = \sec (90° - 13°)$
$= \sec 77°$
$= \dfrac{1}{\cos 77°}$
$= \dfrac{1}{0.225}$
$= 4.444$

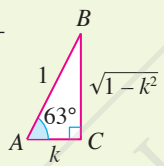
(c) $\cot 13° = \tan (90° - 13°)$
$= \tan 77°$
$= \dfrac{\sin 77°}{\cos 77°}$
$= \dfrac{0.9744}{0.225}$
$= 4.331$

Example 5

Given $\cos 63° = k$, where $k > 0$. Find the value of each of the following in terms of k.
(a) $\sin 63°$ (b) $\sin 27°$ (c) $\operatorname{cosec} 27°$

Solution

(a) $\sin 63° = \sqrt{1 - k^2}$

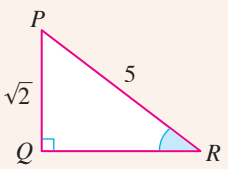


(b) $\sin 27° = \cos (90° - 27°)$
$= \cos 63°$
$= k$

(c) $\operatorname{cosec} 27° = \sec (90° - 27°)$
$= \sec 63°$
$= \dfrac{1}{\cos 63°}$
$= \dfrac{1}{k}$

Self-Exercise 6.2

1. The diagram on the right shows a right-angled triangle PQR. Find the value of each of the following.



(a) $\cot R$ (b) $\sin^2 R$ (c) $\dfrac{\cos R - \sin R}{\operatorname{cosec} R}$

2. Given $\tan \alpha = \dfrac{2}{3}$ and α is an acute angle, find
(a) $\sin \alpha$ (b) $\cos^2 \alpha$ (c) $\cot \alpha$
(d) $\operatorname{cosec} \alpha$ (e) $\dfrac{4 - \sec^2 \alpha}{2 - \sec \alpha}$

3. Find the complementary angles of each of the following.
(a) $54°$ (b) $5° 17' 14''$ (c) $\dfrac{\pi}{5}$ rad

4. Given $\cos 33° = 0.839$ and $\sin 33° = 0.545$, find the value of each of the following.
(a) $\sin 57°$ (b) $\tan 57°$ (c) $\sec 57°$

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Determine the values of the trigonometric ratios for any angle

The values of the trigonometric ratios of any angle can be obtained by using a calculator or any dynamic geometry software. However, there are several methods to determine these trigonometric ratios.

Method 1: Use a calculator

The values of sine, cosine and tangent of any angle can be determined by using a calculator. However, values for cosecant, secant and cotangent of any angle can be calculated by inversing the values of the trigonometric ratios of sine, cosine and tangent of that particular angle.

Information Corner

The use of key depends on the model of the calculator used.

Example 6

Use a calculator and find the value of each of the following trigonometric ratios, correct to four significant figures.

(a) $\sin(-215° 12')$ (b) $\sec(-4.14 \text{ rad})$

DISCUSSION

Discuss how to find the values for the trigonometric ratios when the angles are in radians.

Solution

(a) 0.5764

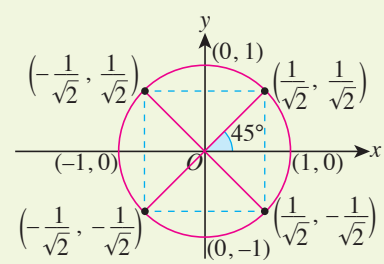
(b) $\sec(-4.14 \text{ rad}) = \dfrac{1}{\cos(-4.14)} = -1.846$

Method 2: Use a unit circle

Example 7

Use the unit circle on the right, and state the values of each of the following.

(a) $\cos 135°$ (b) $\operatorname{cosec}\left(-\frac{\pi}{4} \text{ rad}\right)$



Solution

(a) The coordinates that correspond to 135° are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\cos 135°$ = x-coordinate.

Hence, $\cos 135° = -\frac{1}{\sqrt{2}}$.

(b) The coordinates that correspond to $-\frac{\pi}{4}$ rad are $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = \frac{1}{y\text{-coordinate}}$.

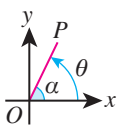
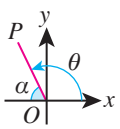
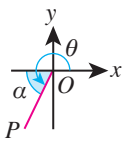
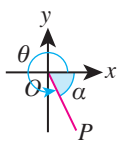
Hence, $\operatorname{cosec}\left(-\frac{\pi}{4}\right) = -\sqrt{2}$.

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Method 3: Use the corresponding trigonometric ratio of the reference angle

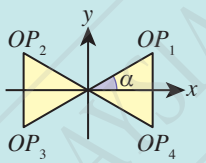
The value of a trigonometric ratio for any angle can be determined by using the trigonometric ratio of the reference angle that corresponds to that angle.

The diagram below shows the reference angles, α for the angles $0° \leqslant \theta \leqslant 360°$ or $0 \leqslant \theta \leqslant 2\pi$.

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
			
$\alpha = \theta$	$\alpha = 180° - \theta$	$\alpha = \theta - 180°$	$\alpha = 360° - \theta$

Information Corner

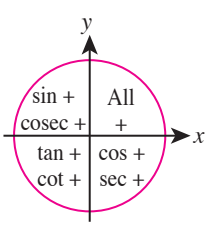
The reference angle, α is an acute angle made by the line OP with the x-axis on a Cartesian plane.



The signs of trigonometric ratios in quadrants I, II, III and IV can be determined by using the coordinates on the unit circle as shown in the table below.

Quadrant	Signs							
	x	y	$\sin\theta = y$	$\cos\theta = x$	$\tan\theta = \frac{y}{x}$	$\text{cosec}\,\theta = \frac{1}{y}$	$\sec\theta = \frac{1}{x}$	$\cot\theta = \frac{x}{y}$
I	+	+	+	+	+	+	+	+
II	–	+	+	–	–	+	–	–
III	–	–	–	–	+	–	–	+
IV	+	–	–	+	–	–	+	–

In conclusion, the sign of each trigonometric ratio of any angle in the different quadrants can be summarised in the diagram on the right.



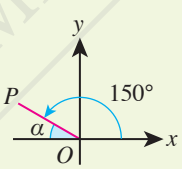
Example 8

Given $\sin 30° = 0.5$ and $\cos 30° = 0.866$, find the value of each of the following.

(a) $\sec 150°$ (b) $\sec\left(-\frac{13}{6}\pi\right)$

Solution

(a)



$\theta = 150°$ is located in Quadrant II.
The sign for sec 150° is negative.
Reference angle, $\alpha = 180° - 150°$
$= 30°$

$\sec 150° = -\sec 30°$
$= -\frac{1}{\cos 30°}$
$= -\frac{1}{0.866}$
$= -1.155$

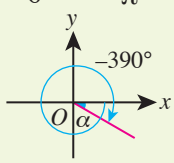
Excellent Tip

Steps to determine the trigonometric ratios without using a calculator.

1. Locate the position of the angle in the quadrant.
2. Determine the sign for the trigonometric ratio.
3. Obtain the corresponding reference angle.
4. Use the trigonometric ratio value of the reference angle.

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(b) $\theta = -\frac{13}{6}\pi \times \frac{180}{\pi} = -390°$



–390° lies in Quadrant IV.
The sign for sec (–390°) is positive.
Reference angle,
$\alpha = 390° - 360°$
$= 30°$

$\sec\left(-\frac{13}{6}\pi\right) = \sec(-390°)$
$= \sec 30°$
$= \frac{1}{\cos 30°}$
$= \frac{1}{0.866}$
$= 1.155$

Flash Quiz

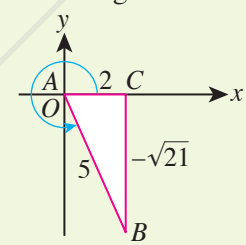
Complete the following trigonometric ratios for the negative angles as the example given.

sin (–A)	–sin A
cos (–A)	
tan (–A)	
cot (–A)	
sec (–A)	
cosec (–A)	

Example 9

Given $\cos A = \frac{2}{5}$ and $270° \leq A \leq 360°$, find the value for each of the following.
(a) tan A (b) sin A (c) sec A

Solution



$BC = \sqrt{5^2 - 2^2} = \sqrt{21}$

(a) $\tan A = -\frac{\sqrt{21}}{2}$ (b) $\sin A = -\frac{\sqrt{21}}{5}$ (c) $\sec A = \frac{5}{2}$

Method 4: Use a right-angled triangle

The trigonometric ratios of special angles 30°, 45° and 60° can be determined by using right-angled triangles. Let explore further into this.

Discovery Activity 3 Group 21st cl

Aim: To determine the trigonometric ratios of special angles by using right-angled triangles

Steps:

1. Diagram 6.3 shows a square while Diagram 6.4 shows an isosceles triangle. Redraw Diagrams 6.3 and 6.4 on a piece of paper.

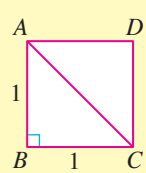


Diagram 6.3

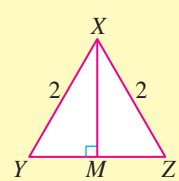


Diagram 6.4

2. Then determine the value of each of the following.
(a) AC (b) YM (c) XM (d) ∠ACB (e) ∠XYZ (f) ∠MXY

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3. Based on Diagram 6.3 or Diagram 6.4, copy and complete the table below.

Angle \ Ratio		sin	cos	tan	cosec	sec	cot
30°	$\frac{\pi}{6}$			$\frac{1}{\sqrt{3}}$	2		
45°	$\frac{\pi}{4}$		$\frac{1}{\sqrt{2}}$			$\sqrt{2}$	
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$					

4. Discuss in groups and briefly present your findings in front of the class.

From the results of Discovery Activity 3, it is found that the trigonometric ratios of the angles, namely 30°, 45° and 60°, are as follows:

Angle \ Ratio		sin	cos	tan	cosec	sec	cot
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$	$\sqrt{3}$
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}$	$\sqrt{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2	$\frac{1}{\sqrt{3}}$

Information Corner

Besides the angles 30°, 45° and 60°, angles 0°, 90°, 180°, 270° and 360° are also special angles.

CHAPTER 6

Example 10

By using the trigonometric ratios of special angles, find the value of each of the following.

(a) cos 315° (b) $\cot\left(\frac{5}{3}\pi\right)$ (c) sec (−480°)

Solution

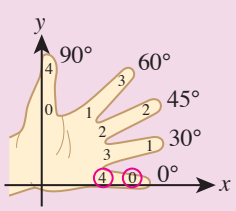
(a) $\cos(315°)$
$= \cos(360° - 315°)$
$= \cos 45°$
$= \frac{1}{\sqrt{2}}$

(b) $\cot\left(\frac{5}{3}\pi\right) = \cot 300°$
$= -\cot(360° - 300°)$
$= -\cot 60°$
$= -\frac{1}{\sqrt{3}}$

(c) $\sec(-480°) = \sec(-480° - (-360°))$
$= \sec(-120°)$
$= -\sec 60°$
$= -2$

Excellent Tip

You can use your fingers to memorise the trigonometric ratio of the special angles.



$\sin 0° = \frac{\sqrt{N}}{2} = \frac{\sqrt{0}}{2} = 0$

$\cos 0° = \frac{\sqrt{N}}{2} = \frac{\sqrt{4}}{2} = 1$

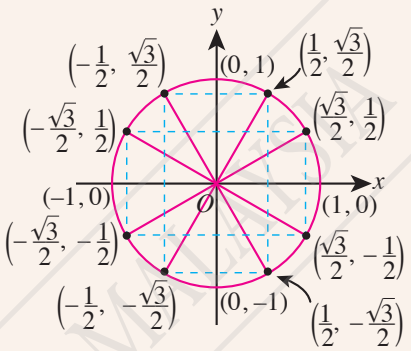
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Self-Exercise 6.3

1. Find the value of each of the following by using a calculator. Give your answers correct to four decimal places.

(a) $\tan 165.7°$ (b) $\cot(-555°)$ (c) $\operatorname{cosec}^2(-1.2 \text{ rad})$ (d) $\sec\left(-\frac{16}{9}\pi\right)$

2. Using the unit circle on the right, find the value of each of the following.



(a) $\sin 330°$ (b) $\tan\left(\frac{2}{3}\pi\right)$

(c) $\cot\left(\frac{7}{6}\pi\right)$ (d) $\cos 600°$

(e) $\operatorname{cosec}\left(-\frac{7}{2}\pi\right)$ (f) $\sin\left(\frac{\pi}{2}\right) - \sec 3\pi$

3. Find the acute angle corresponding to the following angles.

(a) $335°$ (b) $\frac{2}{3}\pi$ rad (c) $\frac{7}{3}\pi$ rad (d) $710°$

4. Using the trigonometric ratios of special angles, find the values of each of the following.

(a) $\sec 150°$ (b) $\operatorname{cosec} 240°$ (c) $\cot 315°$

(d) $\sin 45° + \cos 225°$ (e) $\sec 60° + 2\operatorname{cosec} 30°$ (f) $\sec \pi + \cos\frac{\pi}{2}$

Formative Exercise 6.2

Quiz bit.ly/36Xu8GA

1. Given $\tan x = 3t$ for $0° < x < 90°$, express each of the following in terms of t.

(a) $\cot x$ (b) $\sec(90° - x)$ (c) $\operatorname{cosec}(180° - x)$

2. The angle θ lies in quadrant III and $\tan\theta = 3$. Find the value of each of the following.

(a) $\cot\theta$ (b) $\tan(\pi + \theta)$ (c) $\sin(-\theta)$

3. By using the trigonometric ratios of special angles, find

(a) $2\sin 45° + \cos 585°$ (b) $\tan 210° - \cot(-240°)$

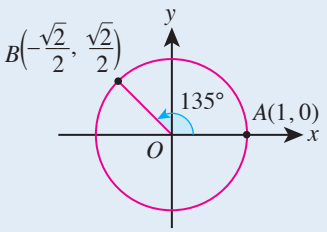
(c) $\operatorname{cosec}\frac{5}{6}\pi + \sin\frac{1}{6}\pi$ (d) $\tan 2\pi - 6\operatorname{cosec}\frac{3}{2}\pi$

4. Without using a calculator, find the value of each of the following.

(a) $\sin 137°$ if $\sin 43° \approx 0.6820$ (b) $\sec 24°$ if $\sec 336° \approx 1.095$

(c) $\tan 224°$ if $\tan 44° \approx 0.9656$ (d) $\cot 15°$ if $\cot 195° \approx 3.732$

5. The diagram on the right shows a unit circle with angle $135°$ marked on it. Based on the information in the unit circle, state the value of each of the following.



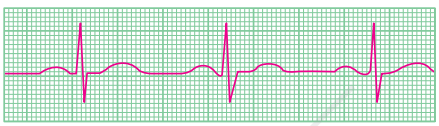
(a) $\sin 135°$ (b) $\sec 135°$

(c) $\cot 45°$ (d) $\operatorname{cosec}(-45°)$

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6.3 Graphs of Sine, Cosine and Tangent Functions

The diagram on the right shows the heartbeat rhythm of a healthy person. This rhythm is known as the Normal Sinus Rhythm. Note that this rhythm is an example of a trigonometric function graph.



The graphs for the trigonometric functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx + c$, where a, b and c are constants and $b > 0$, can be constructed using any dynamic geometric software or just manually using tables of values and graph papers.

Graphs of trigonometric functions

Discovery Activity 4

Group | 21st cl | STEM | CT

Aim: To draw and determine the properties of sine, cosine and tangent graphs

Steps:

1. Form three groups.
2. Then, copy and complete the table below.

$x°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
x rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
$y = \sin x$													
$y = \cos x$													
$y = \tan x$													

3. Using graph papers or any dynamic geometry software, draw the following graphs.
   Group I: $y = \sin x$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.
   Group II: $y = \cos x$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.
   Group III: $y = \tan x$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.
4. After that, copy and complete the table below.

y-intercept	x-intercept	Maximum value of y	Minimum value of y	Amplitude	Period

5. Each group appoints a representative to present the findings to the class.
6. Other members of the group may ask the representative questions.
7. Repeat steps 5 and 6 until all the groups have completed the presentation.

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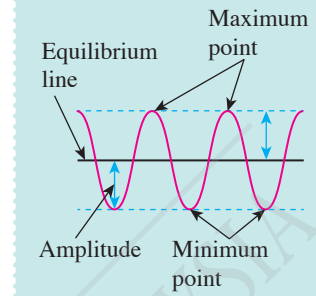
From the results of Discovery Activity 4, it is found that:

- The graphs of $y = \sin x$ and $y = \cos x$ are sinusoidal and have the following properties:

(a) The maximum value is 1 while the minimum value is –1, so the amplitude of the graph is 1 unit.

(b) The graph repeats itself every 360° or 2π rad, so 360° or 2π rad is the period for both graphs.

Information Corner



- The graph $y = \tan x$ is not sinusoidal. The properties of $y = \tan x$ are as follows:

(a) This graph has no maximum or minimum value.

(b) The graph repeats itself every 180° or π rad interval, so the period of a tangent graph is 180° or π rad.

(c) The function $y = \tan x$ is not defined at $x = 90°$ and $x = 270°$. The curve approaches the line $x = 90°$ and $x = 270°$ but does not touch the line. This line is called an asymptote.

DISCUSSION

Discuss the meaning of:
- amplitude
- period
- cycle
- asymptote

The graphs for these three functions are seen to be periodic as the x-domain is extended. Look at the following graph.

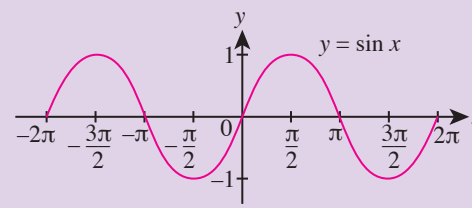
1 Graph $y = \sin x$ for $-2\pi \leq x \leq 2\pi$

(a) Amplitude = 1
  (i) The maximum value of $y = 1$
  (ii) The minimum value of $y = -1$

(b) Period = 360° or 2π

(c) x-intercepts: $-2\pi, -\pi, 0, \pi, 2\pi$

(d) y-intercepts: 0



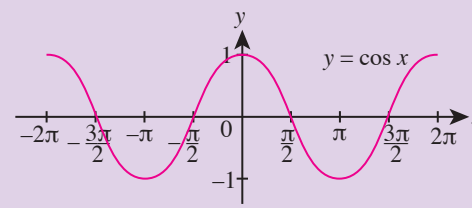
2 Graph $y = \cos x$ for $-2\pi \leq x \leq 2\pi$

(a) Amplitude = 1
  (i) The maximum value of $y = 1$
  (ii) The minimum value of $y = -1$

(b) Period = 360° or 2π

(c) x-intercepts: $-\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi$

(d) y-intercepts: 1



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3 Graph $y = \tan x$ for $-2\pi \leqslant x \leqslant 2\pi$

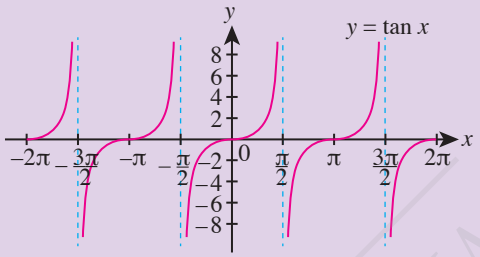
(a) No amplitude
  (i) There is no maximum value of y
  (ii) There is no minimum value of y

(b) Period = 180° or π

(c) x-asymptotes: $-\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi$

(d) x-intercepts: $-2\pi, -\pi, 0, \pi, 2\pi$

(e) y-intercepts: 0



In Discovery Activity 5, you will investigate the effect of different transformation on the graph $y = a \sin bx + c$, $a \neq 0$ and $b > 0$.

Discovery Activity 5

Group | 21st cl | STEM | CT

Aim: Compare sine function graphs of different equation forms

Steps:

1. Copy and complete the following table.

$x°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
x rad	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
$y = \sin x$													
$y = 3 \sin x$													
$y = 3 \sin 2x$													
$y = 3 \sin 2x + 1$													

2. Using a graph paper or any dynamic geometry software, draw each of the following pairs of functions on the same axes.
(a) $y = \sin x$ and $y = 3 \sin x$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.
(b) $y = \sin x$ and $y = 3 \sin 2x$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.
(c) $y = \sin x$ and $y = 3 \sin 2x + 1$ for $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$.

3. Next, compare each pair of graphs in terms of their amplitudes, periods and the position of the graph.

4. Then, draw conclusions on the relationship between the values a, b and c in the function $y = a \sin bx + c$, where $a \neq 0$ and $b > 0$, in terms of
(i) the amplitude,
(ii) the period,
(iii) the position
of the function graph.

5. Each group appoints a representative to present the findings to the class.

6. Other members of the group may ask the representative questions.

KEMENTERIAN PENDIDIKAN MALAYSIA

From the results of Discovery Activity 5, it is found that the values of a, b and c in the function $y = a \sin bx + c$ affect the amplitude, the period and the position of the graph.

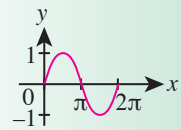
$y = a \sin bx + c$

a

- If $c = 0$:
  Amplitude $= |a|$, Maximum value of $y = a$, Minimum value of $y = -a$
- If $c \neq 0$:
  Amplitude $= |a|$ or
  $\frac{(\text{maximum value} - \text{minimum value})}{2}$

sin

Shape of graph:



b

- Number of cycles in the range $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$
- Period $= \frac{360°}{b}$
  $= \frac{2}{b}\pi$

c

Translation $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the basic graph.

Similar transformations can be done on the graphs $y = \cos x$ and $y = \tan x$. It is found that the original shapes of the graphs remain unchanged. The effects of changing the values of a, b and c on the graph can be summarised in the following table:

Change in	Effects
a	The maximum and minimum values of the graphs (except for the graph of $y = \tan x$ where there is no maximum or minimum value).
b	Number of cycles in the range $0° \leqslant x \leqslant 360°$ or $0 \leqslant x \leqslant 2\pi$: • Graphs $y = \sin x$ and $y = \cos x$ $\left(\text{period} = \frac{360°}{b} \text{ or } \frac{2}{b}\pi\right)$ • Graph $y = \tan x$ $\left(\text{period} = \frac{180°}{b} \text{ or } \frac{1}{b}\pi\right)$
c	The position of the graph with reference to the x-axis as compared to the position of the basic graph

After knowing the shapes and properties of the trigonometric function graphs, two important skills that need to be mastered are drawing and sketching those graphs.

QR Access

- Let's explore the function graph for $y = a \cos (bx - c) + d$.

ggbm.at/p5kyyhym

- Let's explore the function graph for $y = k + A \tan (Bx + C)$.

ggbm.at/kjqc2vcn

Example 11

Draw the graph $y = 3 - 2 \cos \frac{3}{2}x$ for $0 \leqslant x \leqslant 2\pi$.

Solution

To determine the class interval size:

$b = \frac{3}{2}$, Period $= 2\pi \div \frac{3}{2} = \frac{4}{3}\pi$

Class interval size $= \left(\frac{4}{3}\pi\right) \div 8$

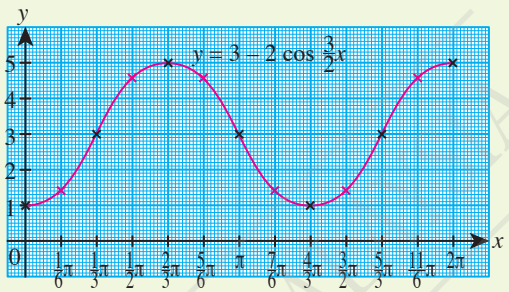
$= \frac{\pi}{6}$

Excellent Tip

To draw a trigonometric function graph, we need at least eight points for one cycle.

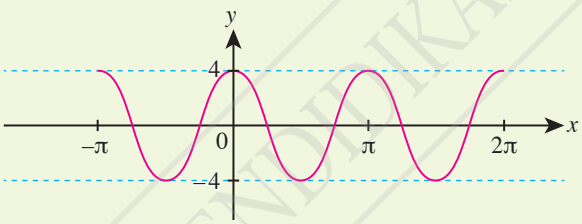
x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2}{3}\pi$	$\frac{5}{6}\pi$	π	$\frac{7}{6}\pi$	$\frac{4}{3}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	$\frac{11}{6}\pi$	2π
$y = 3 - 2\cos\frac{3}{2}x$	1	1.59	3	4.41	5	4.4	3	1.59	1	1.59	3	4.41	5

The graph $y = 2\cos\frac{3}{2}x$ is reflected on the x-axis, then followed by a translation $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.



Example 12

State the cosine function represented by the graph in the diagram below.



Solution

Note that the amplitude is 4.
So, $a = 4$.
Two cycles in the range of $0 \leqslant x \leqslant 2\pi$.
The period is π, that is, $\frac{2\pi}{b} = \pi$, so $b = 2$.
Hence, the graph represents $y = 4\cos 2x$.

Besides identifying the trigonometric function of a given graph, the values of constants a, b and c also help in sketching graphs when the trigonometric functions are given.

Example 13

Given $f(x) = 3\sin 2x$ for $0° \leqslant x \leqslant 360°$.
(a) State the period of the function graph $y = f(x)$. Then, state the number of cycles in the given range.
(b) State the amplitude of the graph.
(c) Write the coordinates of the maximum and minimum points.
(d) Sketch the function graph $y = f(x)$.
(e) On the same axis, sketch the function graph $y = -3\sin 2x$.

Solution

(a) The period of the function graph $y = f(x)$ is $\frac{360°}{2} = 180°$.
The number of cycles is 2.

(b) The amplitude of the graph is 3.

(c) The maximum points are (45°, 3) and (225°, 3) while the minimum points are (–135°, –3) and (–315°, –3).

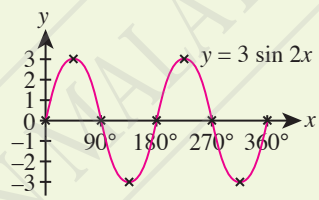
(d) To sketch the function graph $y = 3 \sin 2x$, $0° \leqslant x \leqslant 360°$:

Number of classes $= 2 \times 2 \times 2$
$= 8$

Class interval size $= \frac{360°}{8}$
$= 45°$

x	0°	45°	90°	135°	180°	225°	270°	235°	360°
y	0	3	0	–3	0	3	0	–3	0

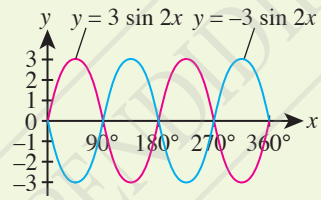
Plot the points: (0, 0), (45°, 3), (90°, 0), (135°, –3), (180°, 0), (225°, 3), (270°, 0), (335°, –3), (360°, 0)



Excellent Tip

To sketch the graph $y = a \sin bx + c$, $0 \leqslant x \leqslant n\pi$:
- Number of classes is $b \times n \times 2 = m$
- Class interval size $= \frac{n\pi}{m}$

(e) Sketch the function graph $y = –3 \sin 2x$ which resembles a reflection of $y = 3 \sin 2x$ on the x-axis.



Example 14

State the transformation on the function graph $y = \tan x$ to obtain each of the following graphs.

(a) $y = |–\tan x|$ (b) $y = –|\tan x|$

Then, sketch both graphs for $0 \leqslant x \leqslant 2\pi$.

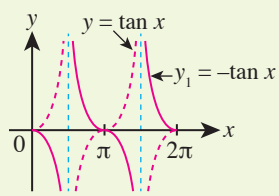
Solution

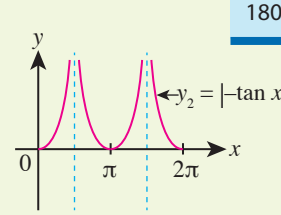
Period $= \pi$ rad

(a) The reflection of the graph $y = \tan x$ on the x-axis results in getting the graph $y_1 = –\tan x$ to be followed by a reflection of the negative part of the graph $y_1 = –\tan x$ on the x-axis to get the graph $y_2 = |–\tan x|$.

Recall

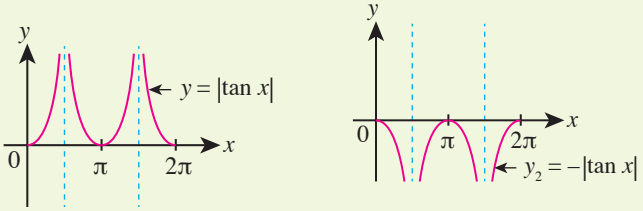
The period for $y = \tan x$ is 180° or π rad.





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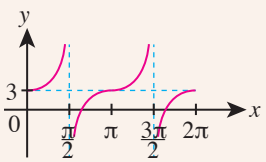
(b) The reflection of the negative part of the graph $y = \tan x$ on the x-axis results in getting the graph $y_1 = |\tan x|$ to be followed by reflection on the x-axis to obtain $y_2 = -|\tan x|$.



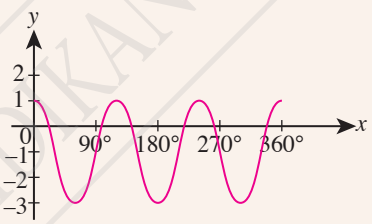
Self-Exercise 6.4

1. Sketch the graph for each of the following functions on a graph paper. Then, check your graphs by using a dynamic geometry software.
(a) $y = 1 - 3 \sin 2x$ for $-90° \leqslant x \leqslant 180°$
(b) $f(x) = -|\tan 2x| + 1$ for $0 \leqslant x \leqslant \pi$

2. State the function represented by each of the following graphs.
(a)



(b)



3. Given $f(x) = A \sin Bx + C$ for $0° \leqslant x \leqslant 360°$. The amplitude of the graph is 3, its period is 90° and the minimum value of $f(x)$ is -2.
(a) State the values of A, B and C.
(b) Sketch the graph of the function.

4. Copy and complete the following table.

Function	Amplitude	Number of cycles/ period	Translation	Sketch the graph $0 \leqslant x \leqslant \pi$
1. $y = \frac{3}{2} \sin 3x$				
2. $y = \lvert\tan 2x\rvert + 1$				

Solving trigonometric equations using graphical method

The solution to a trigonometric equation can be determined by drawing two graphs which are derived from the trigonometric equations in the same diagram. The solutions are the values of x for the coordinates of the points of intersection of the two graphs.

Example 15

On the same axes, draw the graphs $y = \sin 2x$ and $y = \frac{x}{2\pi}$ for $0 \leqslant x \leqslant \pi$. Then, state the solutions to the trigonometric equation $2\pi \sin 2x - x = 0$.

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Solution

For the graph $y = \sin 2x$:

x	0	$\frac{\pi}{8}$	$\frac{\pi}{4}$	$\frac{3\pi}{8}$	$\frac{\pi}{2}$	$\frac{5\pi}{8}$	$\frac{3\pi}{4}$	$\frac{7\pi}{8}$	π
y	0	0.71	1	0.71	0	−0.71	−1	−0.71	0

Range = π
Class interval size = $\frac{\pi}{8}$

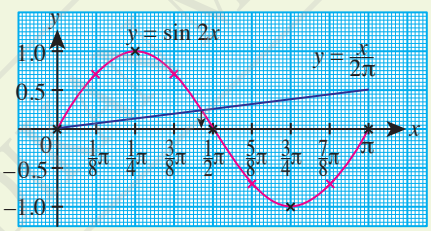
For the straight line $y = \frac{x}{2\pi}$:

x	0	π
y	0	0.5
Point	(0, 0)	(π, 0.5)

The graphs $y = \sin 2x$ and $y = \frac{x}{2\pi}$:

The points of intersection of the two graphs are the solutions to $\sin 2x = \frac{x}{2\pi}$ or $2\pi \sin 2x - x = 0$

From the graph, it is found that the solutions to the equation $2\pi \sin 2x - x = 0$ are 0 and 0.46π.



The number of solutions to a trigonometric equation can be determined by sketching the graphs for the functions involved on the same axes. The number of intersection points will give the number of solutions to the equation.

Example 16

Sketch the graph $y = 3 \cos 2x + 2$ for $0 \leq x \leq \pi$. Then, determine the number of solutions to the following trigonometric equations.

(a) $3x \cos 2x = \pi - 2x$

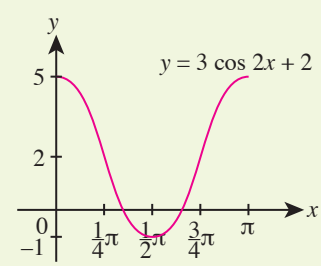
(b) $3\pi \cos 2x = 8x - \pi$

Solution

Given $y = 3 \cos 2x + 2$

Number of classes = $(2 \times 1) \times 2 = 4$

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	5	2	−1	2	5



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(a) To determine the number of solutions for $3x \cos 2x = \pi - 2x$,

$3x \cos 2x + 2x = \pi$

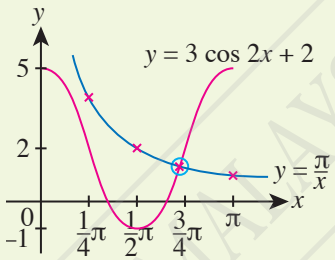
$x(3 \cos 2x + 2) = \pi$

$3 \cos 2x + 2 = \frac{\pi}{x}$

Hence, $y = 3 \cos 2x + 2$ and $y = \frac{\pi}{x}$.

For $y = \frac{\pi}{x}$:

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	π
y	∞	4	2	1
Point	–	$\left(\frac{\pi}{4}, 4\right)$	$\left(\frac{\pi}{2}, 2\right)$	$(\pi, 1)$



Hence, the number of solutions = 1.

(b) To determine the number of solutions for $3\pi \cos 2x = 8x - \pi$,

$3\pi \cos 2x + \pi = 8x$

$\pi(3 \cos 2x + 1) = 8x$

$3 \cos 2x + 1 = \frac{8x}{\pi}$

$3 \cos 2x + 1 + 1 = \frac{8x}{\pi} + 1$.

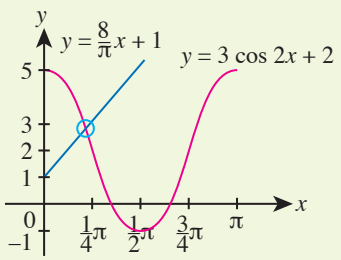
Thus, $y = 3 \cos 2x + 2$ and $y = \frac{8x}{\pi} + 1$.

Excellent Tip

Only two points are needed to sketch a linear function graph.

For $y = \frac{8x}{\pi} + 1$:

x	0	$\frac{1}{4}\pi$
y	1	3
Point	(0, 1)	$\left(\frac{1}{4}\pi, 3\right)$



Hence, the number of solutions = 1.

Self-Exercise 6.5

1. By using appropriate scales,

(a) draw the following graphs for $0° \leqslant x \leqslant 360°$.

(i) $y = \frac{1}{2} \sin 2x$ (ii) $y = 2 - \cos x$ (iii) $y = -\tan 2x + 1$

(b) draw the following graphs for $0 \leqslant x \leqslant 2\pi$.

(i) $y = 3 \cos 2x$ (ii) $y = -3 \sin x + 2$ (iii) $y = |\tan x| - 1$

2. Sketch the graph of the function $y = -2|\sin 2x| + 1$ for $0 \leqslant x \leqslant 2\pi$.

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3. On the same axes, sketch the graphs of function $y = \frac{3}{2}\cos 3x$ and $y = \frac{x}{\pi} + 1$ for $0 \leqslant x \leqslant \frac{\pi}{2}$. Then, state the number of solutions for $3\cos 3x = \frac{2x}{\pi} + 2$ for $0 \leqslant x \leqslant \frac{\pi}{2}$.

4. Determine the number of solutions for $x - 2\pi|\cos 2x| = 0$ for $0 \leqslant x \leqslant \pi$ by sketching two suitable graphs.

Formative Exercise 6.3

Quiz bit.ly/3nDPEWx

1. Using a scale of 2 cm to 0.5 units on the x-axis and y-axis, draw the graph $y = 2\cos\frac{\pi}{2}x$ for $0 \leqslant x \leqslant 4$. From the graph obtained, estimate the values of x that satisfy the equation $\cos\frac{\pi}{2}x + \frac{1}{4} = 0$ for $0 \leqslant x \leqslant 4$.

2. Using a scale of 2 cm to $\frac{\pi}{6}$ rad on x-axis and 1 cm to 1 unit on y-axis, draw the graph $y = 5\tan x$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$. On the same axes, draw a suitable straight line to solve the equation $30\tan x - 6x + 5\pi = 0$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$. Then, find the value of x in radians.

3. Sketch the graph $y = 3\sin 2x$ for $0 \leqslant x \leqslant 2\pi$. Then, on the same axes, draw a suitable straight line to find the number of solutions for the equation $3\pi\sin 2x + 2x = 3\pi$. State the number of solutions.

4. Sketch the graph $y = |\cos 2x|$ for $0 \leqslant x \leqslant \pi$. On the same axes, draw a straight line to find the number of solutions for the equation $x - 2\pi|\cos 2x| = 0$. Then, state the number of solutions.

5. Using a scale of 2 cm to $\frac{\pi}{4}$ rad on the x-axis and 2 cm to 1 unit on the y-axis, draw on the same axes, the graphs of the trigonometric functions $y = 1 + \sin 2x$ and $y = |2\cos 2x|$ for $0 \leqslant x \leqslant 2\pi$. Then, state the coordinates of the points of intersection of the two graphs.

6. By sketching the graph $y = 3 + |\cos x|$ for $0 \leqslant x \leqslant 2\pi$, find the range of values of k such that $|\cos x| = k - 3$ has no real roots.

7. (a) Sketch the graph $y = -2\cos\frac{3x}{2}$ for $0 \leqslant x \leqslant 2\pi$.

(b) Then, by using the same axes, draw a suitable graph to solve the equation $2\cos\frac{3x}{2} + \frac{\pi}{2x} = 0$ for $0 \leqslant x \leqslant 2\pi$. State the number of solutions.

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6.4 Basic Identities

Derive the basic identities

Note the following three basic identities:

$\sin^2\theta + \cos^2\theta = 1$ $\quad 1 + \tan^2\theta = \sec^2\theta$ $\quad 1 + \cot^2\theta = \operatorname{cosec}^2\theta$

A trigonometric identity is an equation that involves trigonometric functions and is valid for any values of angle. Trigonometric identities that we have learnt are as follows:

$$\tan\theta = \frac{\sin\theta}{\cos\theta},\ \cot\theta = \frac{1}{\tan\theta} \text{ and } \operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

By using a unit circle and a right-angled triangle, three more basic identities which are also known as Pythagoras identities can be proven.

Discovery Activity 6 Group 21st cl

Aim: Derive the basic identities

Steps:

1. Divide students into two groups.
2. Group 1 will deal with Diagram 6.5 and Group 2 will deal with Diagram 6.6.

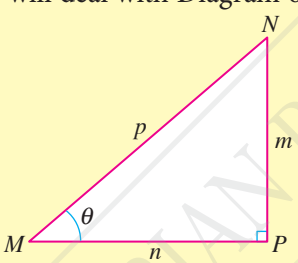


Diagram 6.5

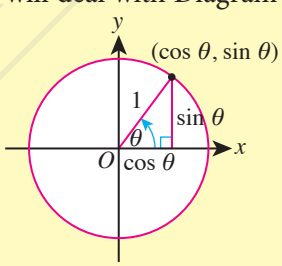
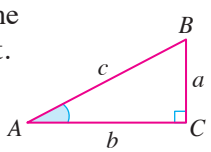


Diagram 6.6

Group 1	Group 2
(a) List the six trigonometric ratios in terms of n, m and p.	(a) Write x in terms of $\cos\theta$ and y in terms of $\sin\theta$.
(b) Using the Pythagoras theorem $m^2 + n^2 = p^2$, derive the three basic identities.	(b) Using the Pythagoras theorem $x^2 + y^2 = 1$, derive the three basic identities.

3. Discuss in your groups and present your findings to the class.

From Discovery Activity 6, it is found that all three basic identities can be derived by using a right-angled triangle ABC and all the trigonometric ratios which have been learnt.



- $\sin A = \frac{a}{c}$, $\operatorname{cosec} A = \frac{c}{a}$
- $\cos A = \frac{b}{c}$, $\sec A = \frac{c}{b}$
- $\tan A = \frac{a}{b}$, $\cot A = \frac{b}{a}$

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By using Pythagoras theorem, it is known that $a^2 + b^2 = c^2$. Divide the two sides of the equation by a^2, b^2 and c^2; we get:

$\div a^2$	$\div b^2$	$\div c^2$
$\frac{a^2}{a^2} + \frac{b^2}{a^2} = \frac{c^2}{a^2}$	$\frac{a^2}{b^2} + \frac{b^2}{b^2} = \frac{c^2}{b^2}$	$\frac{a^2}{c^2} + \frac{b^2}{c^2} = \frac{c^2}{c^2}$
$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$	$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$	$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$
$1 + \cot^2 A = \operatorname{cosec}^2 A$	$1 + \tan^2 A = \sec^2 A$	$\sin^2 A + \cos^2 A = 1$

These three basic trigonometric identities can be used to solve problems involving trigonometric ratios.

Example 17

Without using a calculator, find the value of each of the following.

(a) $\sin^2(-430°) + \cos^2(-430°)$

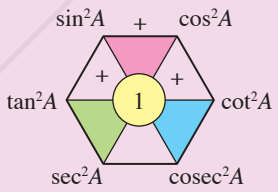
(b) $\tan^2\left(\frac{\pi}{3}\right) - \sec^2\left(\frac{\pi}{3}\right)$

Solution

(a) $\sin^2(-430°) + \cos(-430°) = 1$

(b) $\tan^2\left(\frac{\pi}{3}\right) - \sec^2\left(\frac{\pi}{3}\right) = -1$

Excellent Tip



$\sin^2 A + \cos^2 A = 1$

$1 + \tan^2 A = \sec^2 A$

$1 + \cot^2 A = \operatorname{cosec}^2 A$

Self-Exercise 6.6

1. Without using a calculator, find the value of each of the following.

(a) $\cos^2 80° + \sin^2 80°$

(b) $\sec^2 173° - \tan^2 173°$

(c) $1 - \cos^2 45°$

(d) $\operatorname{cosec}^2 \frac{8}{5}\pi - \cot^2 \frac{8}{5}\pi$

2. Given $\cos\theta = m$, determine the values of the following in terms of m.

(a) $\sec^2\theta$

(b) $\sin^2\theta$

(c) $\cot^2\theta$

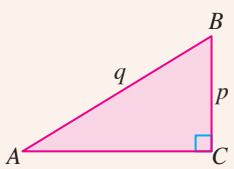
3. It is given that $0 \leqslant \theta \leqslant \frac{\pi}{2}$ and $\tan\theta = 3$. Without using a right-angled triangle, find the values of $\sin\theta$ and $\cos\theta$.

4. The diagram on the right shows a right-angled triangle ABC. Write the following expressions in terms of p and/or q.

(a) $1 - \cos^2 A$

(b) $\operatorname{cosec}^2 A - 1$

(c) $1 - \sec^2 A$



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Prove trigonometric identities by using the basic identities

Example 18

Prove each of the following trigonometric identities.
(a) $1 - 2\sin^2 A = 2\cos^2 A - 1$
(b) $\tan A + \cot A = \sec A \operatorname{cosec} A$

Solution

(a) $1 - 2\sin^2 A$
$= 1 - 2(1 - \cos^2 A)$ ← Use the identity $\sin^2 A + \cos^2 A = 1$
$= 1 - 2 + 2\cos^2 A$
$= 2\cos^2 A - 1$

Hence, it is proven that $1 - 2\sin^2 A = 2\cos^2 A - 1$

(b) $\tan A + \cot A$ ← Use the identity $\tan A = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$
$= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}$
$= \frac{\sin^2 A + \cos^2 A}{\cos A \sin A}$ ← Use the identity $\sin^2 A + \cos^2 A = 1$
$= \frac{1}{\cos A \sin A}$ ← Use the identity $\frac{1}{\sin A} = \operatorname{cosec} A$ and $\frac{1}{\cos A} = \sec A$
$= \sec A \operatorname{cosec} A$

Hence, it is proven that $\tan A + \cot A = \sec A \operatorname{cosec} A$

Excellent Tip

To prove the trigonometric identities:
(a) Prove from the more complex side.
(b) Convert to basic trigonometric ratios form.
(c) Multiply by a conjugate if required.

QR Access

Activities to verify the basic identities using clinometer

bit.ly/2Rq1vIU

Proofs can be done by simplifying the expressions on the left until they are similar to the expressions on the right or vice versa. Proof is also possible by simplifying the expressions on the left and the expressions on the right until both expressions are the same. This method is shown in the example below.

Example 19

Prove that $\tan^2 x - \sec^2 x + 2 = \operatorname{cosec}^2 x - \cot^2 x$.

Solution

Left-hand side: $\tan^2 x - \sec^2 x + 2 = (-1) + 2$
$= 1$ ← Use the identity $1 + \tan^2 x = \sec^2 x$

Right-hand side: $\operatorname{cosec}^2 x - \cot^2 x = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$ ← Use the identity $\frac{1}{\sin x} = \operatorname{cosec} x$ and $\frac{1}{\tan x} = \cot x$
$= \frac{1 - \cos^2 x}{\sin^2 x}$ ← Use the identity $\sin^2 x + \cos^2 x = 1$
$= \frac{\sin^2 x}{\sin^2 x}$
$= 1$

Hence, $\tan^2 x - \sec^2 x + 2 = \operatorname{cosec}^2 x - \cot^2 x = 1$.

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Self-Exercise 6.7

1. Prove each of the following trigonometric identities.

(a) $3 \sin^2 A - 2 = 1 - 3 \cos^2 A$

(b) $1 + 2 \tan^2 A = \dfrac{1 - \sin^4 A}{\cos^4 A}$

(c) $\sec A \operatorname{cosec} A - \tan A = \cot A$

(d) $\cos^2 A - \sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

(e) $\cot^2 \theta - \tan^2 \theta = \operatorname{cosec}^2 \theta - \sec^2 \theta$

(f) $\dfrac{\sin^2 \theta}{1 + \cos \theta} = 1 - \cos \theta$

(g) $\tan^2 \theta(\operatorname{cosec}^2 \theta - 1) = 1$

(h) $\dfrac{1 - 2 \sin^2 \theta}{\cos \theta - \sin \theta} = \cos \theta + \sin \theta$

Formative Exercise 6.4

Quiz bit.ly/3nHaLaI

1. Given $\sec^2 \theta = p$, find the value of each of the following, in terms of p.

(a) $\tan^2 \theta$ (b) $\cos^2 \theta$ (c) $\sin^2 \theta$

2. Without using a calculator, find the value of each of the following.

(a) $\sin^2 100° + \cos^2 100°$

(b) $\tan^2 3 \text{ rad} - \sec^2 3 \text{ rad}$

(c) $1 + \tan^2 120°$

(d) $1 + \cot^2 225°$

3. Prove each of the following.

(a) $\dfrac{\tan^2 x}{1 + \tan^2 x} = \sin^2 x$

(b) $5 \sec^2 x + 4 = 9 \sec^2 x - 4 \tan^2 x$

(c) $\dfrac{\sin \theta}{1 + \cos \theta} + \dfrac{1 + \cos \theta}{\sin \theta} = 2 \operatorname{cosec} \theta$

(d) $\sec^4 \theta - \sec^2 \theta = \tan^4 \theta + \tan^2 \theta$

4. The following equation is true for all values of θ.

$$\frac{1}{1 + \cos \theta} + \frac{1}{1 - \cos \theta} = 2 \operatorname{cosec}^2 \theta$$

(a) Prove the equation.

(b) Then, find the value of $\operatorname{cosec}^2 \theta$ if $\cos \theta = 0.6$.

5. Each of the following identities shows a relation with sec y. Prove each of the following identities.

(a) $\sec y = \sin y \tan y + \cos y$

(b) $\sec y = \dfrac{\tan y + \cot y}{\operatorname{cosec} y}$

(c) $\sec y = \dfrac{1 - \sin y}{2 \cos y} + \dfrac{\cos y}{2 - 2 \sin y}$

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6.5 Addition Formulae and Double Angle Formulae

Proving trigonometric identities using addition formulae

Consider the following example:
$\sin (30° + 60°) = \sin 90° = 1$
However, $\sin 30° + \sin 60° = 0.5 + 0.866 \neq 1$
Hence, $\sin (30° + 60°) \neq \sin 30° + \sin 60°$.
In summary, $\sin (A + B) \neq \sin A + \sin B$.

Information Corner

- Angles in the form $(A + B)$ or $(A - B)$ are called **addition angles**.
- Angles in the form $2A$, $3A$,… are known as **double angles**.

The formulae that are used to find trigonometry ratios of addition angles are as follows:

$$\sin (A + B) = \sin A \cos B + \cos A \sin B$$
$$\sin (A - B) = \sin A \cos B - \cos A \sin B$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B$$
$$\cos (A - B) = \cos A \cos B + \sin A \sin B$$
$$\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$
$$\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

QR Access

To prove addition formulae

bit.ly/32uSYLk

The above formulae are also known as **addition formulae**. Calculator can be used to verify such formulae.

Discovery Activity 7 Group 21st cl

Aim: To verify the addition formulae

Steps:

1. Copy and complete the table below by using a calculator. Besides 10° and 20°, you can select five more sets with any values.

A	B	$\sin (A + B)$	$\sin A \cos B$	$\cos A \sin B$	$\sin A \cos B + \cos A \sin B$
10°	20°				

2. Then, compare the answers obtained in Column 3 with Column 6 in the table above.
3. Discuss your findings with other groups.

From Discovery Activity 7, it is found that one of the addition formulae can be verified, which is $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$. The same method can be used to verify the other addition formulae. Calculator can also be used to verify the examples below.

Example 20

Find the value of each of the following expressions using the addition formulae. Then, check the answers obtained with a calculator.

(a) $\sin 63° \cos 27° + \cos 63° \sin 27°$

(b) $\cos 50° \cos 20° + \sin 50° \sin 20°$

(c) $\dfrac{\tan 70° - \tan 10°}{1 + \tan 70° \tan 10°}$

Solution

(a) $\sin(63° + 27°)$
$= \sin 90°$
$= 1$

(b) $\cos(50° - 20°)$
$= \cos 30°$
$= \dfrac{\sqrt{3}}{2}$

(c) $\tan(70° - 10°)$
$= \tan 60°$
$= \sqrt{3}$

Prove other identities by using the addition formulae

The addition formulae can be used to prove the other trigonometric identities.

Example 21

Prove the following identities.

(a) $\sin(90° + A) = \cos A$

(b) $\sin\left(x + \dfrac{\pi}{6}\right) - \sin\left(x - \dfrac{\pi}{6}\right) = \cos x$

Solution

(a) $\sin(90° + A)$
$= \sin 90° \cos A + \cos 90° \sin A$
$= (1) \cos A + (0) \sin A$
$= \cos A$

(b) $\sin\left(x + \dfrac{\pi}{6}\right) - \sin\left(x - \dfrac{\pi}{6}\right)$
$= \sin x \cos\left(\dfrac{\pi}{6}\right) + \cos x \sin\left(\dfrac{\pi}{6}\right) - \left(\sin x \cos\left(\dfrac{\pi}{6}\right) - \cos x \sin\left(\dfrac{\pi}{6}\right)\right)$
$= \sin x \cos\left(\dfrac{\pi}{6}\right) + \cos x \sin\left(\dfrac{\pi}{6}\right) - \sin x \cos\left(\dfrac{\pi}{6}\right) + \cos x \sin\left(\dfrac{\pi}{6}\right)$
$= 2 \cos x \sin\left(\dfrac{\pi}{6}\right)$
$= 2 \cos x \left(\dfrac{1}{2}\right)$
$= \cos x$

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Use of addition formulae

Let's look at some examples of how to use addition formulae to solve problems involving trigonometric ratios.

Example 22

Without using a calculator, find the values of the following.
(a) sin 105° (b) tan 15°

Recall

	sin	cos	tan
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Solution

(a) $\sin 105°$
$= \sin (45° + 60°)$
$= \sin 45° \cos 60° + \cos 45° \sin 60°$
$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) + \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right)$
$= \left(\frac{1 + \sqrt{3}}{2\sqrt{2}}\right) \times \left(\frac{\sqrt{2}}{\sqrt{2}}\right)$
$= \frac{\sqrt{2} + \sqrt{6}}{4}$

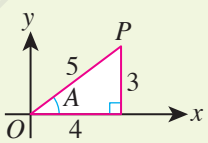
(b) $\tan 15°$
$= \tan (60° - 45°)$
$= \frac{\tan 60° - \tan 45°}{1 + \tan 60° \tan 45°}$
$= \frac{\sqrt{3} - 1}{1 + (\sqrt{3})(1)}$
$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$
$= 2 - \sqrt{3}$

Example 23

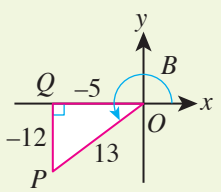
Given $\sin A = \frac{3}{5}$, $0° < A < 90°$ and $\sin B = -\frac{12}{13}$, $90° < B < 270°$. Find
(a) $\sin (A + B)$ (b) $\tan (B - A)$

Solution

(a) $\sin (A + B) = \sin A \cos B + \cos A \sin B$
$= \left(\frac{3}{5}\right)\left(\frac{-5}{13}\right) + \left(\frac{4}{5}\right)\left(\frac{-12}{13}\right)$
$= \frac{-15 - 48}{65}$
$= -\frac{63}{65}$



(b) $\tan (B - A) = \frac{\tan B - \tan A}{1 + \tan B \tan A}$
$= \frac{\left(\frac{-12}{-5}\right) - \left(\frac{3}{4}\right)}{1 + \left(\frac{-12}{-5}\right)\left(\frac{3}{4}\right)}$
$= \frac{\left(\frac{48 - 15}{20}\right)}{1 + \left(\frac{36}{20}\right)}$
$= \left(\frac{33}{20}\right) \times \left(\frac{20}{56}\right)$ ← $\frac{33}{20} \div \frac{56}{20} = \frac{33}{20} \times \frac{20}{56}$
$= \frac{33}{56}$



Excellent Tip

Based on the diagram in Example 23:
- $\sin A = \frac{3}{5}$, $\sin B = \frac{-12}{13}$
- $\cos A = \frac{4}{5}$, $\cos B = \frac{-5}{13}$
- $\tan A = \frac{3}{4}$, $\tan B = \frac{12}{5}$

Flash Quiz

Based on Example 23, determine the values of the following:
(a) cosec $(A + B)$
(b) sec $(A - B)$
(c) cot $(B - A)$

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Self-Exercise 6.8

1. Prove each of the following trigonometric identities.

(a) $\sin(x-y) - \sin(x+y) = -2\cos x \sin y$ (b) $\tan\left(A + \frac{\pi}{4}\right) = \frac{1+\tan A}{1-\tan A}$

(c) $\frac{\cos(x-y) - \cos(x+y)}{\sin(x+y) + \sin(x-y)} = \tan y$ (d) $\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}$

2. Without using a calculator, find the value of each of the following.

(a) $\cos 75°$ (b) $\operatorname{cosec} 105°$ (c) $\cot 195°$

3. Given $\cos x = -\frac{5}{13}$ for $0 < x < \pi$ and $\sin y = -\frac{3}{5}$ for $\frac{\pi}{2} < y < \frac{3}{2}\pi$, find the value of each of the following.

(a) $\sin(x+y)$ (b) $\cos(x-y)$ (c) $\cot(x+y)$

Deriving the double angle formulae

The addition formulae can be used to derive double-angle formulae.

sin 2A

- Given $\sin(A+B) = \sin A \cos B + \cos A \sin B$
- If B is substituted with A,
  $\sin(A+A) = \sin A \cos A + \cos A \sin A$
  Hence, $\mathbf{\sin 2A = 2 \sin A \cos A}$

cos 2A

- Given $\cos(A+B) = \cos A \cos B - \sin A \sin B$
- If B is substituted with A,
  $\cos(A+A) = \cos A \cos A - \sin A \sin A$.
  Hence, $\mathbf{\cos 2A = \cos^2 A - \sin^2 A}$
- If we substitute $\sin^2 A = 1 - \cos^2 A$ into $\cos 2A = \cos^2 A - \sin^2 A$,
  $\cos 2A = \cos^2 A - (1 - \cos^2 A)$
  $= 2\cos^2 A - 1$
  Hence, $\mathbf{\cos 2A = 2\cos^2 A - 1}$
- If we substitute $\cos^2 A = 1 - \sin^2 A$ into $\cos 2A = \cos^2 A - \sin^2 A$,
  $\cos 2A = (1 - \sin^2 A) - \sin^2 A$
  $= 1 - 2\sin^2 A$
  Hence, $\mathbf{\cos 2A = 1 - 2\sin^2 A}$

tan 2A

- Given $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$
- If B is substituted with A,
  $\tan(A+A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$
  Hence, $\mathbf{\tan 2A = \frac{2\tan A}{1 - \tan^2 A}}$

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Example 24

Find the value of each of the following expressions using the double-angle formulae. Then, verify the answers obtained with a calculator.

(a) $2 \sin 15° \cos 15°$ (b) $\cos^2 22.5° - \sin^2 22.5°$ (c) $\dfrac{2 \tan 75°}{1 - \tan^2 75°}$

Solution

(a) $2 \sin 15° \cos 15°$
$= \sin 2(15°)$
$= \sin 30°$
$= \dfrac{1}{2}$

(b) $\cos^2 22.5° - \sin^2 22.5°$
$= \cos 2(22.5°)$
$= \cos (45°)$
$= \dfrac{\sqrt{2}}{2}$

(c) $\dfrac{2 \tan 75°}{1 - \tan^2 75°}$
$= \tan 2(75°)$
$= \tan 150°$
$= -\dfrac{1}{\sqrt{3}}$

Proving trigonometric identities using double-angle formulae

Example 25

Prove the following identities.

(a) $\text{cosec } 2A = \dfrac{1}{2} \sec A \text{ cosec } A$

(b) $\cos \theta - \sin \theta = \dfrac{\cos 2\theta}{\cos \theta + \sin \theta}$

Solution

(a) Given $\text{cosec } 2A = \dfrac{1}{2} \sec A \text{ cosec } A$

Prove: Left-hand side $= \text{cosec } 2A$

$= \dfrac{1}{\sin 2A}$ ← Use the identity $\text{cosec } 2A = \dfrac{1}{\sin 2A}$

$= \dfrac{1}{2 \sin A \cos A}$

$= \dfrac{1}{2} \sec A \text{ cosec } A$ ← Use the identity $\dfrac{1}{\sin A} = \text{cosec } A$ and $\dfrac{1}{\cos A} = \sec A$

(b) Given $\cos \theta - \sin \theta = \dfrac{\cos 2\theta}{\cos \theta + \sin \theta}$

Prove: Right-hand side $= \dfrac{\cos 2\theta}{\cos \theta + \sin \theta}$

$= \dfrac{(\cos^2 \theta - \sin^2 \theta)}{\cos \theta + \sin \theta} \times \dfrac{(\cos \theta - \sin \theta)}{(\cos \theta - \sin \theta)}$

$= \dfrac{(\cos^2 \theta - \sin^2 \theta)(\cos \theta - \sin \theta)}{(\cos^2 \theta - \sin^2 \theta)}$ ← Use the identity $\cos 2\theta = \cos^2 \theta - \sin^2 \theta$ and multiply by its conjugate

$= \cos \theta - \sin \theta$

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Other formulae involving double angles can be derived by induction. For example if $\cos 2A = 2\cos^2 A - 1$, hence the formula is $\cos 4A = 2\cos^2 2A - 1$. By using the similar method, it is found that $\cos A = 2\cos^2 \frac{A}{2} - 1$. This relation can be used to prove half-angle formulae where $\sin \frac{A}{2}$, $\cos \frac{A}{2}$ and $\tan \frac{A}{2}$ are expressed in terms of $\sin A$ and $\cos A$ as stated below.

- $\sin \frac{A}{2} = \pm\sqrt{\frac{1 - \cos A}{2}}$
- $\cos \frac{A}{2} = \pm\sqrt{\frac{1 + \cos A}{2}}$
- $\tan \frac{A}{2} = \pm\sqrt{\frac{\sin A}{1 + \cos A}}$

Information Corner

- $\sin A = 2\sin \frac{A}{2}\cos \frac{A}{2}$
- $\cos A = \cos^2 \frac{A}{2} - \sin^2 \frac{A}{2}$
  $= 2\cos^2 \frac{A}{2} - 1$
  $= 1 - 2\sin^2 \frac{A}{2}$
- $\tan A = \frac{2\tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

Example 26

Prove that $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$.

Solution

$$\text{Right-hand side} = \frac{1 - \cos x}{\sin x}$$

$$= \frac{1 - \left(1 - 2\sin^2 \frac{x}{2}\right)}{2\sin \frac{x}{2}\cos \frac{x}{2}}$$

Use $\cos 2x = 1 - 2\sin^2 x$
hence, $\cos x = 1 - 2\sin^2 \frac{x}{2}$

$$= \frac{2\sin^2 \frac{x}{2}}{2\sin \frac{x}{2}\cos \frac{x}{2}}$$

$$= \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}}$$

$$= \tan \frac{x}{2}$$

Hence, it is proven that $\tan \frac{x}{2} = \frac{1 - \cos x}{\sin x}$.

DISCUSSION

Prove that:

- $\sin^2 \frac{\theta}{2} = \frac{1 - \cos \theta}{2}$
- $\cos^2 \frac{\theta}{2} = \frac{1 + \cos \theta}{2}$
- $\tan^2 \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}$

Self-Exercise 6.9

1. Without using a calculator, determine the value of each of the following.

(a) $2\sin 30° \cos 30°$ (b) $\cos^2 165° - \sin^2 165°$ (c) $\frac{1 - \tan^2 75°}{2\tan 75°}$

2. Prove that $\operatorname{cosec} 2A = \frac{1}{2}\sec A \operatorname{cosec} A$.

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3. Prove each of the following identities.

(a) $\sin 2\theta(\tan \theta + \cot \theta) = 2$

(b) $\dfrac{\sin 4x + \sin 2x}{\cos 4x + \cos 2x + 1} = \tan 2x$

(c) $\operatorname{cosec} 2A + \cot 2A = \cot A$

(d) $\sec 2x = \dfrac{\cot x + \tan x}{\cot x - \tan x}$

4. Given $\sin x = \dfrac{4}{5}$ where x is an acute angle and $\sin y = \dfrac{5}{13}$ where y is an obtuse angle, find

(a) $\operatorname{cosec} 2x$ (b) $\sec 2y$ (c) $\sin \dfrac{x}{2}$ (d) $\tan \dfrac{y}{2}$

Formative Exercise 6.5

Quiz bit.ly/34MeLhn

1. Given $\tan (A + B) = 3$ and $\tan B = \dfrac{1}{3}$, find the value of $\tan A$.

2. Given that $3A = 2A + A$, prove each of the following by using the suitable identities.

(a) $\sin 3A = 3 \sin A - 4 \sin^3 A$

(b) $\cos 3A = 4 \cos^3 A - 3 \cos A$

CHAPTER 6

3. Given that $\sin x = \dfrac{24}{25}$ for $0 \leqslant x \leqslant \dfrac{\pi}{2}$ and $\cos y = \dfrac{8}{17}$ for $\pi \leqslant y \leqslant 2\pi$, find

(a) $\cos (x + y)$ (b) $\operatorname{cosec} (x - y)$ (c) $\tan (x - y)$

(d) $\sec 2y$ (e) $\sin \dfrac{y}{2}$

4. Prove each of the following identities.

(a) $\cot (x + y) = \dfrac{\cot x \cot y - 1}{\cot x + \cot y}$

(b) $\tan y = \dfrac{\cos (x - y) - \cos (x + y)}{\sin (x - y) + \sin (x + y)}$

5. Given $\tan \theta = t$ for $0 \leqslant \theta \leqslant \pi$. Express each of the following in terms of t.

(a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$

(d) $\sin^2 \dfrac{\theta}{2}$ (e) $\cos^2 \dfrac{\theta}{2}$

6. Prove each of the following identities.

(a) $\tan \dfrac{1}{2}\theta = \dfrac{\sin \theta}{1 + \cos \theta}$ (b) $\sec^2 \dfrac{1}{2}\theta = \dfrac{2}{1 + \cos \theta}$ (c) $\sin 2\theta = \dfrac{2 \tan \theta}{1 + \tan^2 \theta}$

7. By using the addition identities, show that

(a) $\tan \left(\theta + \dfrac{\pi}{2}\right) = -\cot \theta$ (b) $\cos \left(\theta + \dfrac{\pi}{2}\right) = -\sin \theta$ (c) $\sin \left(\theta + \dfrac{\pi}{2}\right) = \cos \theta$

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6.6 Trigonometric Function Applications

Solving trigonometric equations

Consider the following question:

> Given $\sin \theta = 0.5$, what is the value of θ?

- The value of θ can be obtained by using the $\sin^{-1} 0.5$ function in the calculator, that is, $\sin^{-1} 0.5 = 30°$.
- It is found that the values of sin 150°, sin 390°, sin 510°, … are also 0.5. Hence, the angles 150°, 390°, 510°, … are also the solutions of $\sin \theta = 0.5$.
- If the range for the angles is not stated, then the number of solutions for a trigonometric equation will be infinite.

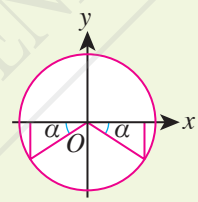
To solve a trigonometric equation, knowledge of the trigonometric identities, the reference angle and the sign of the trigonometric ratio in a quadrant are important.

Example 27

Solve the following equations for $0° \leqslant \theta \leqslant 360°$.
(a) $\sin \theta = -0.5446$ (b) $\cos 2\theta = 0.3420$

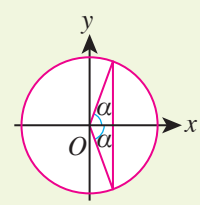
Solution

(a) $\sin \theta = -0.5446$
Reference angle, $\alpha = \sin^{-1}(0.5446)$
$\alpha = 33°$



$\sin \theta$ is negative, so θ is in the quadrant III and IV for $0° \leqslant \theta \leqslant 360°$.
$\theta = 180° + 33°$ and $360° - 33°$
$= 213°$ and $327°$

(b) $\cos 2\theta = 0.3420$
Reference angle, $\alpha = \cos^{-1}(0.3420)$
$\alpha = 70°$



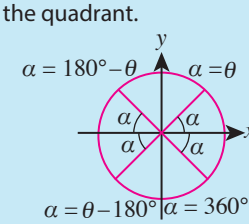
$\cos 2\theta$ is positive, so 2θ is in the quadrant I and IV for $0° \leqslant 2\theta \leqslant 720°$
$2\theta = 70°, 360° - 70°, 360° + 70°$ and $360 + (360° - 70°)$
$= 70°, 290°, 430°$ and $650°$
$\theta = 35°, 145°, 215°$ and $325°$

Excellent Tip

Steps to solve a trigonometric equation:
1. Simplify the equation by using suitable identities if needed.
2. Determine the reference angle, and use the value of the trigonometric ratio without taking into consideration the signs.
3. Find the angles in the quadrants that correspond to the signs of the trigonometric ratio and range.
4. Write the solutions obtained.

Recall

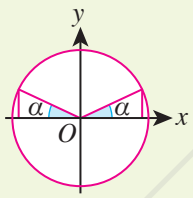
Given α is the reference angle and θ is the ange in the quadrant.



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Example 28

Solve the equation $3 \sin\left(A + \frac{\pi}{3}\right) = 0.99$ for $0 \leq A \leq \pi$.



Solution

$3 \sin\left(A + \frac{\pi}{3}\right) = 0.99$

$\sin\left(A + \frac{\pi}{3}\right) = 0.33$

Reference angle, $\alpha = \sin^{-1}(0.33)$ ← Change the calculator to radian mode

$= 0.3363$ rad

$\sin\left(A + \frac{\pi}{3}\right)$ is positive, so $\left(A + \frac{\pi}{3}\right)$ are in quadrants I and II

for $\frac{\pi}{3} \leq A + \frac{\pi}{3} \leq 4.189$.

$A + \frac{\pi}{3} = 0.3363$ and $\pi - 0.3363$

$A = 0.3363 - \frac{\pi}{3}$ and $2.805 - \frac{\pi}{3}$

$= -0.7109$ and 1.758

Hence, $A = 1.758$ rad.

Excellent Tip

If using the calculator in degree mode:
$\sin^{-1}(0.33) = 19.27°$
Change to radian mode:
$19.27° \times \frac{\pi}{180}$
$= 0.3363$ rad

Example 29

Find the values of x that range from 0° to 360° that satisfy the following equations.
(a) $\sin 2x + \cos x = 0$
(b) $2 \cos 2x - 13 \sin x + 10 = 0$

Solution

(a) $\sin 2x + \cos x = 0$ ← Use the identity $\sin 2x = 2 \sin x \cos x$

$2 \sin x \cos x + \cos x = 0$

$\cos x (2 \sin x + 1) = 0$

So, $\cos x = 0$ or $2 \sin x + 1 = 0$

When $\cos x = 0$,

$x = 90°$ and $x = 270°$

When $2 \sin x + 1 = 0$

$\sin x = -0.5$

Reference angle, $\alpha = 30°$

$\sin x$ is negative, so x is in the quadrant III and IV

$x = 180° + 30°$ and $360° - 30°$

$= 210°$ and $330°$

Hence, $x = 90°, 210°, 270°$ and $330°$.

Flash Quiz

Given $0° \leq x \leq 360°$.
Complete the table below.

Ratio	x
$\sin x = 0$	
$\cos x = 0$	
$\tan x = 0$	
$\sin x = 1$	
$\cos x = 1$	
$\tan x = 1$	
$\sin x = -1$	
$\cos x = -1$	
$\tan x = -1$	

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(b) $2\cos 2x - 13\sin x + 10 = 0$

$2(1 - 2\sin^2 x) - 13\sin x + 10 = 0$ ← $\cos 2x = 1 - 2\sin^2 x$

$2 - 4\sin^2 x - 13\sin x + 10 = 0$

$4\sin^2 x + 13\sin x - 12 = 0$

$(4\sin x - 3)(\sin x + 4) = 0$

$\sin x = 0.75$ or $\sin x = -4$ (ignore) ← $0 \leqslant \sin x \leqslant 1$

When $\sin x = 0.75$, reference angle, $\alpha = 48.59°$

$\sin x$ is positive, so x is in the quadrant I and II.

Hence, $x = 48.59°$ and $131.41°$.

Self-Exercise 6.10

1. Given that $0° \leqslant x \leqslant 360°$, find all the values of x that satisfy each of the following equations.

(a) $\sin 2x = -0.4321$

(b) $\sec(2x + 40°) = 2$

(c) $\cot\left(\frac{x}{3}\right) = 0.4452$

(d) $5\tan x = 7\sin x$

(e) $\sin^2 x - 2\sin x = \cos 2x$

(f) $\sin(x + 30°) = \cos(x + 120°)$

(g) $7\sin x + 3\cos 2x = 0$

(h) $\sin x = 3\sin 2x$

(i) $\cos(x - 60°) = 3\cos(x + 60°)$

2. Find all the angles between 0 and 2π that satisfy the following equations.

(a) $\sin\left(2x + \frac{\pi}{6}\right) = -\frac{\sqrt{3}}{2}$

(b) $3\sin y = 2\tan y$

(c) $3\cot^2 z - 5\operatorname{cosec} z + 1 = 0$

(d) $\sin 2A - \cos 2A = 0$

(e) $\cos B \sin B = \frac{1}{4}$

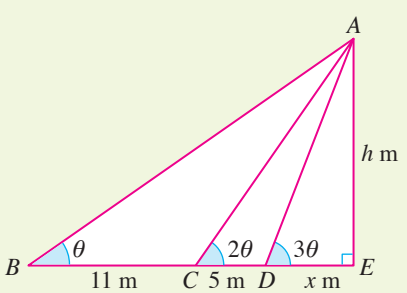
(f) $4\sin(x - \pi)\cos(x - \pi) = 1$

Solving problems involving trigonometric functions

The knowledge of trigonometric functions is often used to solve problems in our daily lives as well as in problems involving trigonometry.

Example 30 MATHEMATICAL APPLICATIONS

In the diagram on the right, AE represents the height of a building. The angles of elevation of A from points B, C and D are θ, 2θ dan 3θ respectively. The points B, C, D and E lie on a horizontal straight line. Given $BC = 11$ m and $CD = 5$ m. If $AE = h$ m and $DE = x$ m, find the height of the building, in terms of x.



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Solution

1 . Understanding the problem

- Given $BC = 11$ m, $CD = 5$ m, $DE = x$ m with angles θ, 2θ, and 3θ.
- Find the height of the building, $AE = h$ m.

2 . Planning the strategy

- Find $\tan\theta$, $\tan 2\theta$ and $\tan 3\theta$, in terms of h and x.
- Use the identity $\tan 3\theta = \tan(\theta + 2\theta)$.
- Substitute the expressions for $\tan\theta$, $\tan 2\theta$ and $\tan 3\theta$.
- Simplify the equation to find h.

3 . Implementing the strategy

It is found: $\tan\theta = \dfrac{h}{16+x}$

$\tan 2\theta = \dfrac{h}{5+x}$

$\tan 3\theta = \dfrac{h}{x}$ where $\tan 3\theta = \tan(\theta + 2\theta)$.

$$\frac{h}{x} = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta\tan 2\theta}$$

$$= \frac{\left(\frac{h}{16+x}\right) + \left(\frac{h}{5+x}\right)}{1 - \left(\frac{h}{16+x}\right)\left(\frac{h}{5+x}\right)}$$

$$= \frac{\frac{h(5+x) + h(16+x)}{(16+x)(5+x)}}{\frac{(16+x)(5+x) - h^2}{(16+x)(5+x)}}$$

$$= \frac{h(5+x) + h(16+x)}{(16+x)(5+x) - h^2}$$

So, $\dfrac{1}{x} = \dfrac{21+2x}{80+21x+x^2-h^2}$

$80 + 21x + x^2 - h^2 = x(21 + 2x)$

$80 + 21x + x^2 - h^2 = 21x + 2x^2$

$h^2 = 80 - x^2$

$h = \pm\sqrt{80 - x^2}$

Hence, the height of the building is $\sqrt{80 - x^2}$ m.

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4 . Check and reflect

Let x be 4 m. Then, $h = \sqrt{80 - 4^2}$
$= 8$ m

It is found: $\tan\theta = \frac{8}{20}$
$= \frac{2}{5}$

$\tan 2\theta = \frac{8}{9}$

$\tan 3\theta = \frac{8}{4}$
$= 2$

$\tan 3\theta = \frac{\tan\theta + \tan 2\theta}{1 - \tan\theta \tan 2\theta}$
$= \frac{\left(\frac{2}{5}\right) + \left(\frac{8}{9}\right)}{1 - \left(\frac{2}{5}\right)\left(\frac{8}{9}\right)}$
$= \frac{\left(\frac{18 + 40}{45}\right)}{\left(\frac{45 - 16}{45}\right)}$
$= \frac{58}{29}$
$= 2$

Self-Exercise 6.11

1. In planning a flight, a pilot is required to determine the ground speed, v kmh^{-1}, together with the speed and direction of the wind. The ground speed, in kmh^{-1}, is expressed as

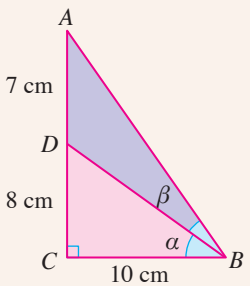
$$v = \frac{770 \sin 135°}{\sin\theta}$$

Without using a calculator, find the value of v, if $\tan\theta = 7$ and $0° < \theta < 180°$.



2. By using the identity $\sec^2 A - \tan^2 A = 1$, find the exact value of $\tan A$ if $\sec^2 A + \tan^2 A = 2$.

3. Elly intends to paste the wallpaper by using a collage technique. The diagram on the right shows a triangle ABC which is made up of two types of coloured paper. The point D is on AC, where $AD = 7$ cm, $DC = 8$ cm, $BC = 10$ cm and $\angle ACB = 90°$. To avoid wastage, Elly needs to get the accurate sizes of the coloured papers. Find the value of each of the following.
(a) $\tan(\alpha + \beta)$ (b) $\tan\alpha$ (c) $\tan\beta$
Then, state the values of α, β, $\angle BAC$, $\angle ADB$ and $\angle BDC$, the length of BD and the length of AB.



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Formative Exercise 6.6

Quiz bit.ly/34S4BM2

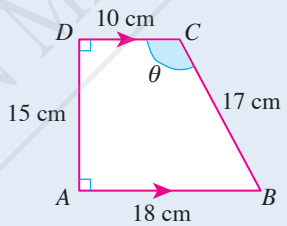
1. Solve each of the following trigonometric equations for $0° \leqslant x \leqslant 360°$.
(a) $2\cos(x - 10°) = -1$ (b) $\tan^2 x = \sec x + 2$ (c) $3\sin x + 4\cos x = 0$

2. Given $0 \leqslant A \leqslant \pi$, solve each of the following equations.
(a) $\sin 2A = \sin 4A$ (b) $5\cot^2 A - 4\cot A = 0$

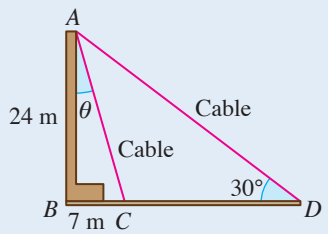
3. Show that $\tan\theta + \cot\theta = \sec\theta \operatorname{cosec}\theta$. Then, solve the equation $\sec\theta \operatorname{cosec}\theta = 4\cot\theta$ for $0° \leqslant x \leqslant 360°$.

4. If A, B and C are angles in the triangle ABC, prove that
(a) $\sin(B + C) = \sin A$, (b) $\cos(B + C) = -\cos A$.

5. The diagram on the right shows a trapezium $ABCD$. The side AB is parallel to DC and $\angle BCD = \theta$. Find the value of each of the following.
(a) $\cos\theta$
(b) $\sin 2\theta$
(c) $\tan 2\theta$
Then, determine the value of θ.

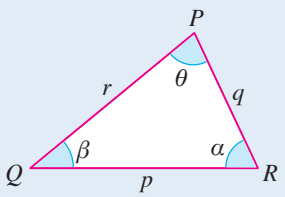


6. An electric pole is reinforced by two cables as shown in the diagram on the right. It is given that the height of the pole, $AB = 24$ m, distance $BC = 7$ m, $\angle BAC = \theta$ and $\angle ADB = 30°$.
(a) Without finding $\angle CAD$, calculate the value of $\sin\angle CAD$, $\cos\angle CAD$ and $\tan\angle CAD$.
(b) State the lengths of the two cables.



7. The diagram on the right shows a triangle PQR with sides p, q and r respectively and the corresponding opposite angles θ, β and α. Show that the area of the triangle is given by the following formula.

$$L = \frac{p^2 \sin\beta \sin\alpha}{2\sin(\beta + \alpha)}$$



8. Given $\sec\theta = t$, where $0 < \theta < \frac{\pi}{2}$. Find the value of each of the following, in terms of t.
(a) $\sin\theta$ (b) $\cos\left(\frac{\pi}{2} + \theta\right)$ (c) $\tan(\pi - \theta)$

9. Sketch the graph of the function $f(x) = 1 + |\cos x|$ for the domain $0 \leqslant x \leqslant 2\pi$.
(a) State the range that corresponds to the domain.
(b) Then, by sketching suitable graphs on the same axes, state the number of solutions for $x|\cos x| = 1 - x$.

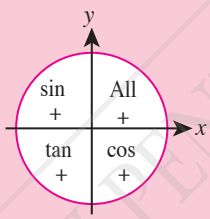
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REFLECTION CORNER

TRIGONOMETRIC FUNCTIONS

Represent positive angles and negative angles in a Cartesian plane
- Angles in degrees or radians.
- Angle in a full circle is 360°.

Determine the trigonometric ratios of any angle:
- Six trigonometric functions
- Reference angle
- Signs for the trigonometric ratios in the 4 quadrants



- Draw and sketch graphs of trigonometric functions.
- Effects of changing a, b and c on the following graphs:
  $y = a \sin bx + c$
  $y = a \cos bx + c$
  $y = a \tan bx + c$
- Find the solutions and determine the number of solutions.

Trigonometric identities
- Complementary angle formulae
- Basic identities
- Addition formulae
- Double angle formulae
- Half angle formulae

Applications

Journal Writing

By using a suitable graphic illustration, produce a summary of all the concepts contained in this chapter. Then, compare your summary with your friends and make improvements if needed. Present your work to the class. Teacher and friends can ask you questions.

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Summative Exercise

1. Write the range of angles for each of the following in radians. PL 1

(a) $0° \leqslant x \leqslant 360°$ (b) $-180° \leqslant x \leqslant 90°$ (c) $270° \leqslant x \leqslant 720°$

2. Write the range of angles for each of the following angles in radians. PL 1

(a) Acute angle (b) Obtuse angle (c) Reflex angle

3. State all the angles for θ between 0° and 360° with the following trigonometric ratios. PL 2

(a) sin θ is 0.66 and –0.66 (b) sec θ is 2.2727 and –2.2727

(c) cot θ is 1.136 and –1.136

4. Without using a calculator, find the value of each of the following. PL 2

(a) sin (–120°) (b) tan 480° (c) sec 750°

(d) cosec 3π (e) cot $\left(-\frac{9}{4}\pi\right)$ (f) cos $\left(-\frac{8}{3}\pi\right)$

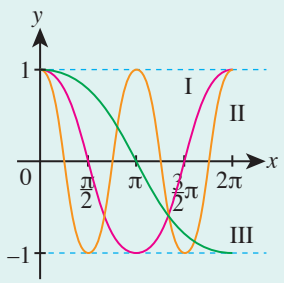
5. Given sin $A = \frac{5}{13}$ and sin $B = \frac{4}{5}$, find the value of cos $(A - B)$ and tan $(A + B)$ if PL 3

(a) A and B are acute angles,

(b) A and B are obtuse angles,

(c) cos A and cos B are negative.

6. The diagram on the right shows three graphs for $y = a \cos bx$ for $0 \leqslant x \leqslant 2\pi$. Copy and complete the table below. PL 3



Graph	Equation	Number of cycles	Period	Class interval
I				
II				
III				

7. (a) State the period of the graph $y = \sin 2x$.

(b) State the amplitude of the graph $y = 1 + 2 \cos 3x$. Then, state the maximum value and the minimum value of y.

(c) On the same axes, sketch each of the following functions for $0 \leqslant x \leqslant \pi$.

(i) $y = \sin 2x$ (ii) $y = 1 + 2 \cos 3x$

(d) State the number of solutions for $\sin 2x - 2 \cos 3x - 1 = 0$ for $0 \leqslant x \leqslant \pi$. PL 3

8. Given a triangle ABC, show that $\sin (A - B) \sin C = \sin^2 A - \sin^2 B$. PL 4

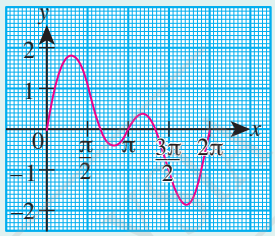
9. Prove the following statement. PL 4

$\tan A \tan (A - 60°) + \tan A \tan (A + 60°) + \tan (A - 60°) \tan (A + 60°) = -3$

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10. Given: $A = \cos^{-1}\left(\frac{3}{\sqrt{10}}\right)$ and $B = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$. If A and B are acute angles, show that $A + B = \frac{\pi}{4}$. PL 4

11. The diagram on the right shows the graph $y = \sin 2x + \sin x$ for $0 \leqslant x \leqslant 2\pi$. PL 4



(a) Find the x-intercept for the graph.

(b) By using the same axis, sketch the graph $y = \cos 2x + 1$. State the maximum value and the period of the graph.

(c) Next, state the number of solutions to the equation $\sin 2x + \sin x = 2\cos^2 x$ in $0 \leqslant x \leqslant 2\pi$.

12. (a) Prove that $\frac{1 - \tan^2 x}{1 + \tan^2 x} = \cos 2x$. PL 4

(b) Sketch the graph of the function $y = \cos 2x$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$.

(c) By using the same axes, draw a suitable straight line to find the number of solutions to the equation $5\pi(1 - \tan^2 x) = x(1 + \tan^2 x)$ for $0 \leqslant x \leqslant \frac{3}{2}\pi$.

13. (a) Solve each of the following trigonometric equations for $0° \leqslant x \leqslant 360°$. PL 5

(i) $\sin(x + 30°) = 2\cos x$

(ii) $2\sec(x + 60°) = 5\sec(x - 20°)$

(iii) $\frac{\tan x + \tan 15°}{1 - \tan x \tan 15°} = 2$

(b) Solve each of the following trigonometric equations for $0 \leqslant x \leqslant 2\pi$.

(i) $3\sin x = 2\cos\left(x + \frac{\pi}{4}\right)$

(ii) $2\tan x + 3\tan\left(x - \frac{\pi}{4}\right) = 0$

(iii) $\tan 5x = \tan 2x$

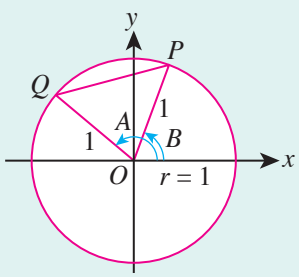
14. The gravitational acceleration is the acceleration due to the gravitational attraction on the body to the centre of the earth. The acceleration, g is dependent on the latitude, θ of the place. The value g can be calculated by using the following formula. PL 5

$$g = 9.78039(1 + 0.005288\sin\theta - 0.000006\sin^2 2\theta)$$

(a) Calculate the gravitational acceleration for Kuala Lumpur.

(b) Determine the latitude when the gravitational acceleration is maximum and state the value.

15. The diagram on the right shows the point $P(\cos B, \sin B)$ and point $Q(\cos A, \sin A)$ located at the circumference of a unit circle with centre O. By using two different methods, find the area of the triangle OPQ. Then, show that $\sin(A - B) = \sin A\cos B - \cos A\sin B$. PL 6



[Hint: Use $\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$ and $\frac{1}{2}ab\sin C$]

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16. The table below shows three non-matching pairs of trigonometric identities. By using any dynamic geometry software, plot each graph to find the matching pairs.

[Hint: Plot $y = \dfrac{1}{\tan x + \cot x}$, $y = \cos^2 x - \sin^2 x$ etc]. PL 6

Left-Hand Side	Right-Hand Side
(a) $\dfrac{1}{\tan x + \cot x} =$	$\cos^2 x - \sin^2 x$
(b) $(\sin x - \cos x)(\tan x + \cot x) =$	$\sin x \cos x$
(c) $\dfrac{\cot x - \tan x}{\cot x + \tan x} =$	$\sec x - \operatorname{cosec} x$

Then, prove each of the identity pairs obtained.

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MATHEMATICAL EXPLORATION

Diagram (a) shows the Magic Hexagon or Super Hexagon which can assist in remembering the various trigonometric identities. Diagram (b) is an example of a reciprocal trigonometric function which is derived from the Magic Hexagon.

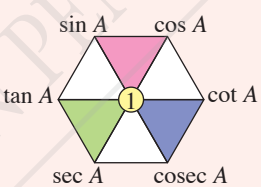
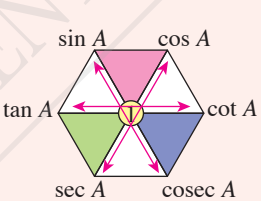


Diagram (a)

Reciprocal Function



$\sin A = \dfrac{1}{\operatorname{cosec} A} \leftrightarrow \operatorname{cosec} A = \dfrac{1}{\sin A}$

$\cos A = \dfrac{1}{\sec A} \leftrightarrow \sec A = \dfrac{1}{\cos A}$

$\tan A = \dfrac{1}{\cot A} \leftrightarrow \cot A = \dfrac{1}{\tan A}$

Diagram (b)

Browse through the Internet to know more about how to generate formulae from the Magic Hexagon. Explain the method used to get these formulae and list all the fomulae generated.

CHAPTER 7

LINEAR PROGRAMMING



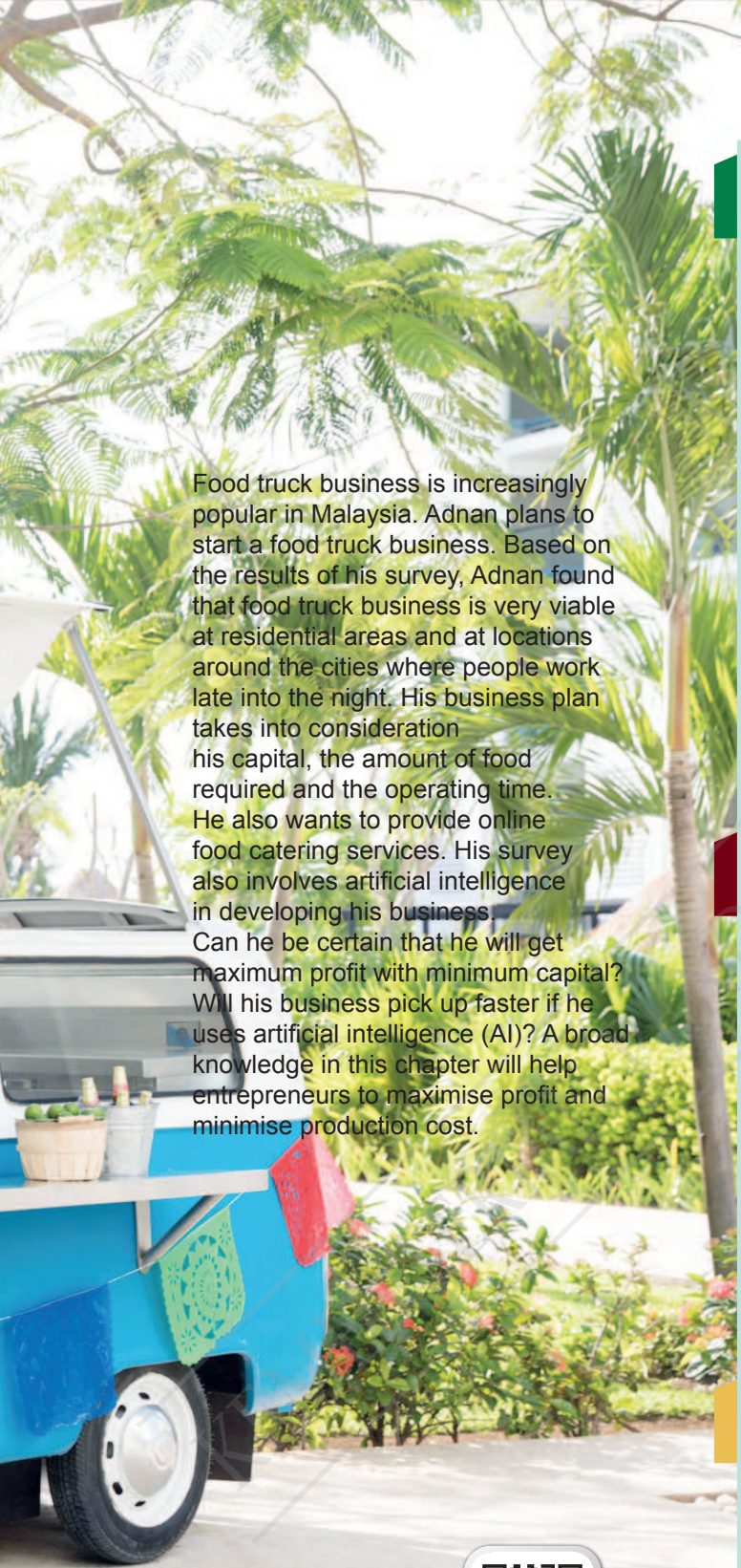
What will be learnt?

- Linear Programming Model
- Application of Linear Programming

List of Learning Standards

bit.ly/3gVApUc

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Food truck business is increasingly popular in Malaysia. Adnan plans to start a food truck business. Based on the results of his survey, Adnan found that food truck business is very viable at residential areas and at locations around the cities where people work late into the night. His business plan takes into consideration his capital, the amount of food required and the operating time. He also wants to provide online food catering services. His survey also involves artificial intelligence in developing his business. Can he be certain that he will get maximum profit with minimum capital? Will his business pick up faster if he uses artificial intelligence (AI)? A broad knowledge in this chapter will help entrepreneurs to maximise profit and minimise production cost.

Video about artificial intelligence (AI)

bit.ly/2YQ1Kjo

Info Corner

George Bernard Dantzig (1914–2005) was an American mathematician who is well known for his contribution in industrial engineering, operations research, computer science, economics and statistics.

He is known for applying algorithm progress to solve linear programming problems.

For more info:

bit.ly/3hZI2KW

Significance of the Chapter

- Linear programming is used widely in ecology, transportation and event organisers to minimise cost and maximise profit.
- Computer software experts use linear programming to solve problems involving thousands of variables and constraints on daily basis.
- Managers of firms use linear programming to plan and make decisions based on resources available.

Keywords

Mathematical model	*Model matematik*
Constraint	*Kekangan*
Objective function	*Fungsi objektif*
Feasible region	*Rantau tersaur*
Optimisation	*Pengoptimuman*

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7.1 Linear Programming Model

Usually, linear programming problems are related to distribution of resources which are limited, such as money, manpower, raw materials and so on, in the best way possible so as to minimise costs or maximise profits.

A linear programming model can be formulated by following the steps given below:

1. Identify the decision variables

Decision variables describe the decisions that need to be made and can be represented by x and y.

2. Identify the objective function

An objective function is a function that needs to be maximised or minimised.

3. Identify the constraints

Present the existing constraints in the form of equations or linear inequalities, which use symbols like $=$, $<$, $\leqslant$, $>$ and/or $\geqslant$. Constraints must be in terms of the decision variables.

QR Access

There are four methods to solve linear programming problems, namely graphical method, simplex method, M method and two-phase method. The most common method used is graphical method. Scan the QR code for information on other methods.

bit.ly/2FNCVPP

What is the most suitable method to solve a linear programming problem that has only two decision variables?

Formulating a mathematical model for a situation based on the given constraints and presenting it graphically

You have learnt linear inequalities in one and two variables. How do you present inequality $y < 4$ or $x \geqslant 2$ graphically? Diagram 7.1 and Diagram 7.2 show the inequality graphs for $y < 4$ and $x \geqslant 2$ respectively.

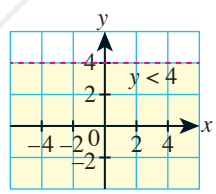


Diagram 7.1

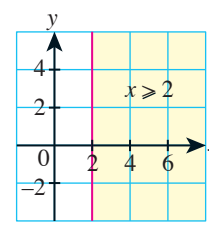


Diagram 7.2

A mathematical model consisting of constraints or objective functions can be obtained from the situation or problem given. Can the mathematical model be illustrated graphically especially in the form of a graph? Let's explore this together.

Discovery Activity 1 Group 21st cl

Aim: To formulate a mathematical model for a situation based on the given constraints and to represent the model graphically

bit.ly/2ZPgpwV

Steps:

1. Scan the QR code on the right or visit the link below it.
2. As a group, select one of the situations in the attachment given. Next, discuss the situation and identify all the constraints. What is a mathematical model?
3. Then, construct a mathematical model in the form of a linear inequality in two variables taking into account all the constraints found.
4. Using GeoGebra software, draw a graph for the linear inequality.
5. Make a conclusion about the position of the shaded region and the type of lines for the graph.

From Discovery Activity 1, it is found that a mathematical model can be formulated by using the variables x and y with the constraints in each situation being $\leqslant$, $\geqslant$, $<$ or $>$.

DISCUSSION

The region which satisfies the inequality $10x - 15y \leqslant 100$ is below the straight line $10x - 15y = 100$. Is this statement true? Discuss.

The region above the straight line $ax + by = c$ satisfies the inequalities $ax + by \geqslant c$ and $ax + by > c$ while the region below the straight line $ax + by = c$ satisfies the inequalities $ax + by \leqslant c$ and $ax + by < c$, where $b > 0$.

The region on the right side of the line $ax = c$ satisfies the inequalities $ax \geqslant c$ and $ax > c$ whereas the region on the left side of the line satisfies the inequalities $ax \leqslant c$ and $ax < c$.
In general, if a mathematical model involves signs like:

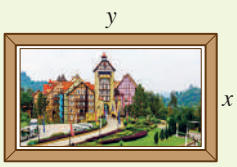
- $\geqslant$ or $\leqslant$, then a solid line (——) is used.
- $<$ or $>$, then a dotted line (------) is used.

CHAPTER 7

Example 1

Write a mathematical model for each of the following situations.
(a) The perimeter of the rectangular photo frame must not be more than 180 cm.
(b) A hawker sells spinach and mustard leaves. The selling prices of 1 kg of spinach and 1 kg of mustard leaves are RM3.50 and RM4.50 respectively. The total sales of the hawker is at least RM350 a day.

Solution



(a) Suppose x and y are the width and length of the rectangular photo frame.
Then, $2x + 2y < 180$.
(b) Suppose x and y are the number of kilograms of spinach and mustard leaves sold in a day respectively. Then, $3.50x + 4.50y \geqslant 350$.

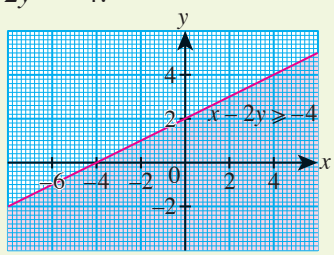
Example 2

Present the following inequalities graphically.

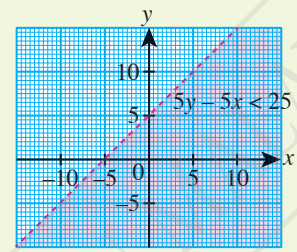
(a) $x - 2y \geqslant -4$ (b) $5y - 5x < 25$

Solution

(a) Given $x - 2y > -4$
Since $b = -2$ (< 0)
Hence, the region lies below the line $x - 2y = -4$.



(b) Given $5y - 5x < 25$
Since $b = 5$ (> 0)
Hence, the region lies below the line $5y - 5x = 25$.



Example 3

Mr Andy plans to build two types of houses, A and B on a plot of land measuring 10 000 m^2. After making a survey, he found out that one unit of house A requires 100 m^2 of land and one unit of house B requires 75 m^2. Mr Andy has a limited land, so the number of houses to be built is at least 200.

(a) Identify the constraints in the problem.
(b) Write a mathematical model to represent the problem.
(c) Draw a graph to represent the mathematical model obtained in (b).

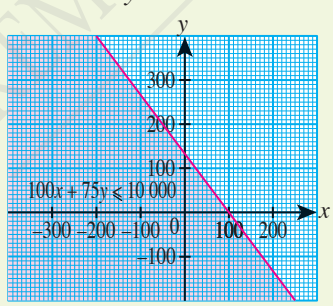
Solution

Let x and y represent houses of types A and B.

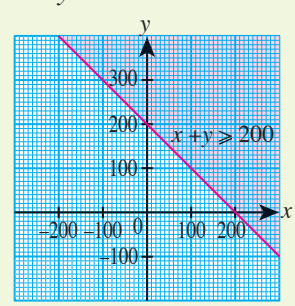
(a) The land area owned by Mr Andy is 10 000 m^2.
The number of houses to be built is at least 200.

(b) Constraint I: $100x + 75y \leqslant 10\,000$
Constraint II: $x + y \geqslant 200$

(c) Constraint I:
$100x + 75y \leqslant 10\,000$



Constraint II:
$x + y \geqslant 200$



Alternative Method

From the graph, for constraint I:

- Select any point for example (100, 200) which lies above the line $100x + 75y = 10\,000$.
  Substitute the coordinates into the inequality $100x + 75y \leqslant 10\,000$.
  $100(100) + 75(200) \leqslant 10\,000$
  $25\,000 \leqslant 10\,000$ (False)
  Hence, the shaded region lies below the line.
- Select any point for example (–200, 200) which lies below the line $100x + 75y = 10\,000$.
  Substitute the coordinates into the inequality $100x + 75y \leqslant 10\,000$.
  $100(-200) + 75(200) \leqslant 10\,000$
  $-5\,000 \leqslant 10\,000$ (True)
  Hence, the shaded region lies below the line.

KEMENTERIAN PENDIDIKAN MALAYSIA

Optimisation in linear programming

Suppose a cake shop makes x chocolate cakes and y cheese cakes costing RM4.00 and RM5.00 respectively. Then, the total cost of making x chocolate cakes and y cheese cakes is $4x + 5y$. Note that $4x + 5y$ is a linear expression. If we want to determine the minimum value of $4x + 5y$, then this linear expression is known as an **objective function**.

In general,

An objective function is written as $k = ax + by$

Discovery Activity 2

Group 21st cl

Aim: To explore how to optimise the objective function

Steps:

1. Scan the QR code on the right or visit the link below it.
2. Drag the slider P left and right. Note the changes that occur on the line d when P moves.
3. Then determine the maximum value in the region.
4. It is given that the objective function is $P = 60x + 90y$. In your respective groups, discuss how to find the maximum value of P in a given region defined by the mathematical model with the following constraints.
   I: $x + y \leqslant 320$ II: $x + 2y \leqslant 600$ III: $5x + 2y \leqslant 1\,000$
5. Present your group's findings to the class and also discuss with other groups.

ggbm.at/ket9dk6r

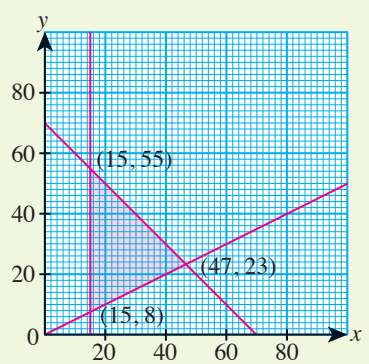
From Discovery Activity 2, it is found that the optimum value of the objective function can be obtained by moving the objective function line parallel to itself towards and into the region that satisfies all the constraints. The optimum value is obtained by substituting the coordinates of the maximum point in the region into the objective function.

Example 4

The diagram on the right shows the shaded region that satisfies a few constraints of a situation.

(a) By using a suitable value of k, draw a line $k = x + 2y$ on the graph. On the same graph, draw a straight line parallel to the line $k = x + 2y$ that passes through each point of the vertices of the region.

(b) Then, find
   (i) the maximum value of $x + 2y$,
   (ii) the minimum value of $x + 2y$.

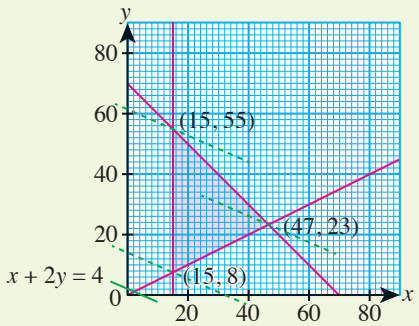


KEMENTERIAN PENDIDIKAN MALAYSIA

Solution

Given $k = x + 2y$.

(a) Let $k = 4$, then $x + 2y = 4$.



(b) (i) Substitute the maximum point for the shaded region, which is (15, 55) into $k = x + 2y$.

$k = 15 + 2(55)$

$k = 125$

Therefore, the maximum value of k is 125.

(ii) Substitute the minimum point for the shaded region, which is (15, 8) into $k = x + 2y$.

$k = 15 + 2(8)$

$k = 31$

Therefore, the minimum value of k is 31.

Excellent Tip

Steps to determine the suitable value of k for $k = ax + by$:

1. Note that a and b are coefficients of x and y respectively.
2. Find the common multiples of a and b.
3. Take k as the common multiple.

Self-Exercise 7.1

1. Graphically illustrate each of the following linear inequalities.

(a) $2y - 3x \geqslant 12$ (b) $6x - y \geqslant 12$ (c) $y + 7x - 49 \leqslant 0$

2. Write a mathematical model based on the following situations.

A car manufacturer produces two types of cars, namely car M and car N. On a given day, the company produces x units of car M and y units of car N.

(a) The number of car N produced is not more than three times the number of car M produced.

(b) The total number of cars produced is at most 80 units.

(c) The number of car N produced is at least 10 units.

3. Consider the situation below. Then answer each of the following questions.

Xin Tian wants to plant banana and papaya trees on a large plot of land of 80 hectares. He hires 360 workers with a capital of at least RM24 000. He uses x hectares of land to plant banana trees and y hectares of land to plant papaya trees. Every hectare planted with banana trees will be supervised by 3 workers while 6 workers will supervise every hectare of papaya trees. The cost to maintain the banana trees is RM800 per hectare while to maintain a hectare of papaya trees is RM300.

(a) Identify the constraints in the above problem.

(b) Form a mathematical model related to the problem above.

(c) Represent each mathematical model obtained in (b) graphically.

KEMENTERIAN PENDIDIKAN MALAYSIA

4. The diagram on the right shows the shaded region which satisfies a few constraints of a situation.

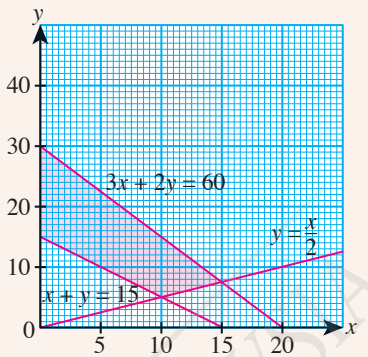
(a) By using a suitable value of k, draw the line $k = x + 2y$ on the graph.

(b) On the same graph, draw straight lines parallel to the line $k = x + 2y$ obtained in (a) to pass through each of the vertices of the region.

(c) Then, find

(i) the maximum value of $x + 2y$,

(ii) the minimum value of $x + 2y$.

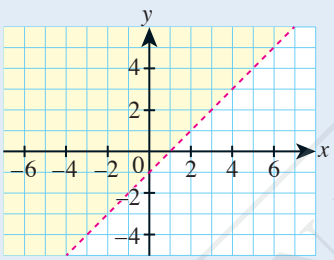


Formative Exercise 7.1

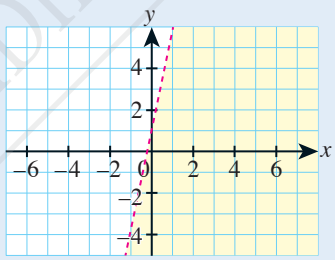
Quiz bit.ly/34MIF53

1. Write an inequality that describes each of the following shaded regions.

(a)



(b)



2. A college offers two academic courses, P and Q. Admission to the college for these courses is based on the following constraints.

I The number of students shall not exceed 100.

II The number of students in course Q is not more than four times the number of students in course P.

III The number of students in course Q exceeds the number of students in course P by at least five people.

Write a mathematical model based on the above situation if x represents the number of students taking course P and y represents the number of students taking course Q.

3. Madam Laili receives a monthly salary of RM3 000. She spends RMx on transport and RMy on food. The monthly expenses on food is at most three times the monthly expenses on transport. The monthly food expenses is at least RM50 more than the monthly expenses on transport. The total monthly expenses on transport and food do not exceed one-third of her monthly salary. Write a mathematical model based on this situation.

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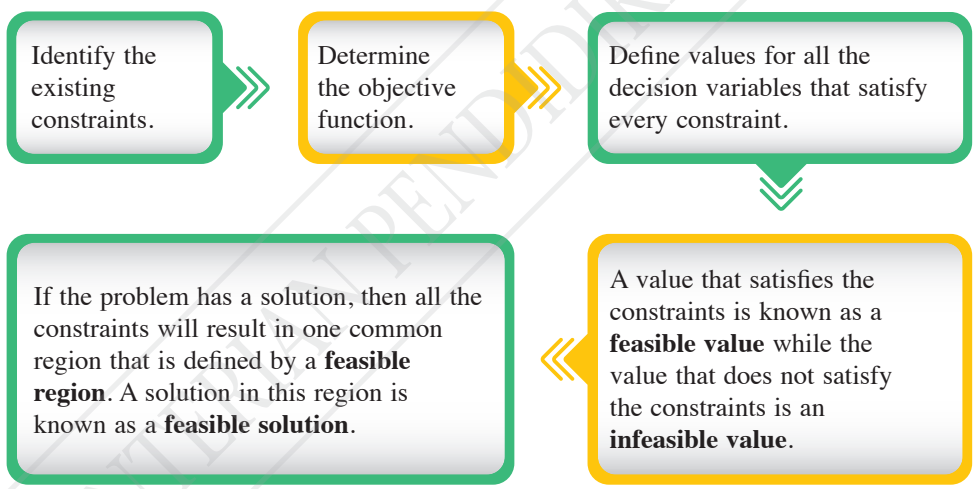
7.2 Linear Programming Applications

In the field of business, businessmen need to make decisions on how to minimise costs and maximise profit. The decisions made are dependent on the existing constraints. How do they solve these problems wisely?

Knowledge in linear programming is important in solving these problems. Through linear programming, we can interpret a problem in terms of its variables. A system of inequalities or linear equations involving those variables can be formed based on the existing conditions or constraints.

Solving problems involving linear programming graphically

Linear programming problems can be solved by drawing graphs of all the related linear equations according to the following steps.



Example 5

A trader wants to arrange x bouquet of roses and y bouquet of orchids. The time taken to arrange a bouquet of roses is 20 minutes while a bouquet of orchids takes 30 minutes. The process of arranging the bouquet of flowers must be based on the following constraints.

I The number of bouquet of orchids must not be more than twice the number of bouquet of roses.

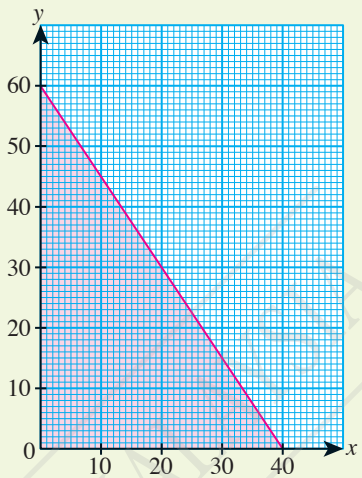
II The number of bouquet of orchids must be at least $\frac{1}{4}$ of the number of bouquet of roses.

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(a) Write a mathematical model involving a system linear inequalities to represent the constraint I and the constraint II.

(b) The third constraint which is represented by the pink region in the diagram is the time taken to arrange the bouquet of flowers. Write the constraint in words.

(c) Construct and label the region R that satisfies the three constraints above. Then, using the same graph, find
   (i) the minimum number of bouquet of orchids if the number of bouquet of roses is 30,
   (ii) the maximum profit of the trader if the profits for each bouquet of roses and each bouquet of orchids are RM35 and RM25 respectively.



Solution

(a) Constraint I: $y \leq 2x$

Constraint II: $y \geq \frac{1}{4}x$

Information Corner

Maximum or optimum points are points at the vertices of a feasible region that can give the optimum value of the objective function.

(b) Consider the points (0, 60) and (40, 0).

The gradient of the straight line, $m = \frac{60 - 0}{0 - 40} = -\frac{3}{2}$

The equation of the straight line,

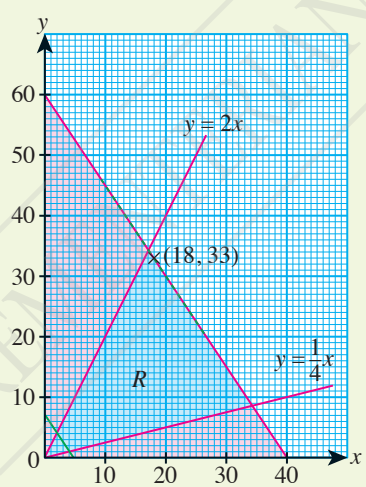
$$y - 0 = -\frac{3}{2}(x - 40)$$

$$2y + 3x = 120$$

$$20y + 30x = 1\,200$$

Therefore, the total time taken to arrange the bouquet of flowers is at least 2 hours.

(c)



(i) Substitute $x = 30$ into $y = \frac{1}{4}x$,

$$y = \frac{1}{4}(30)$$

$$y = 7.5$$

Therefore, the minimum number of bouquet of orchids is 8.

(ii) The maximum point in the shaded region is (18, 33).

Substitute the coordinates of the maximum point into

$k = 35x + 25y$,

$k = 35(18) + 25(33)$

$k = 630 + 825$

$k = 1\,455$

Therefore, the maximum profit made by the trader is RM1 455.

KEMENTERIAN PENDIDIKAN MALAYSIA

Example 6 MATHEMATICAL APPLICATIONS

A school wants to buy two types of tables, P and Q to equip a computer lab. The prices for a table P and a table Q are RM200 and RM100 respectively. The surface area of table P is 1 m^2 while that of table Q is 2 m^2. The school intends to buy x units of table P and y units of table Q. The purchase of the tables will be based on the following constraints.

I The total surface area of the tables is not less than 30 m^2.
II The amount allocated is RM6 000.
III The number of table Q is at most twice that of table P.

(a) Other than $x \geq 0$ and $y \geq 0$, write three linear inequalities that satisfy all the above constraints.
(b) Using a scale of 2 cm to 10 tables on both the x-axis and the y-axis, construct and label the R region that satisfies all of the above constraints.
(c) Based on the graph drawn in (b), find
  (i) the range for the number of tables P if the number of tables Q purchased is 10,
  (ii) the maximum number of pupils who can use the tables at a time if a table P can accommodate 4 pupils and a table Q can accommodate 8 pupils.

Excellent Tip

Problems in a situation can be simplified into tabular form. Based on Example 6, the problem in the situation can be simplified as follows:

	Table P	Table Q
Price	RM200	RM100
Area	1 m^2	2 m^2

Solution

1 . Understanding the problem

- The price of a table P is RM200.
- The price of a table Q is RM100.
- The surface area of each table P is 1 m^2.
- The surface area of each table Q is 2 m^2.
- The total allocation is RM6 000.
- The total surface area of the tables is not less than 30 m^2.
- The maximum number of table Q is twice the number of table P.

2 . Planning the strategy

- Let x be the number of table P and y be the number of table Q.
- The total price for table P is RM200x.
- The total price for table Q is RM100y.

Excellent Tip

Method of solving linear equation problem.

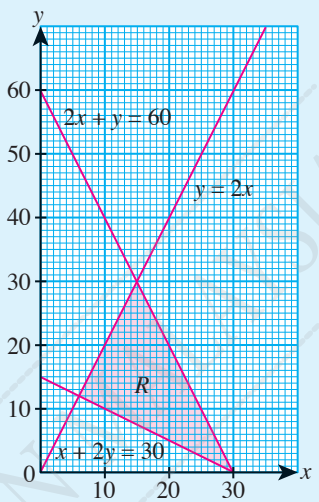
1. Interpret the problem and determine the variables.
2. Define a mathematical model in terms of a system of linear inequalities.
3. Draw graphs and determine the feasible region, R.
4. Write the objective function for the quantity you want to maximise or minimise, that is $k = ax + by$.
5. Select a suitable value for k and draw the straight line.

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3 . Implementing the strategy

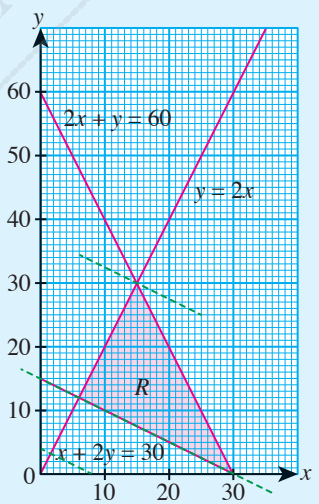
(a) Constraint I:
$x + 2y \geqslant 30$
Constraint II:
$200x + 100y \leqslant 6\,000$
$2x + y \leqslant 60$
Constraint III:
$y \leqslant 2x$
Therefore, the three linear inequalities that satisfy all the constraints are $x + 2y \geqslant 30$, $2x + y \leqslant 60$ and $y \leqslant 2x$.

(b)



(c) (i) Given that the number of table Q to be purchased is 10. Then, draw a straight line $y = 10$.
From the graph, the straight line $y = 10$ intersects the region with the minimum value of $x = 10$ and the maximum value of $x = 25$.
Therefore, the range of the number of table P is $10 \leqslant x \leqslant 25$.

(ii) Let the maximum number of pupils using tables P and Q be $k = 4x + 8y$.
Let $k = 4 \times 8 = 32$.
From the graph, it is found that the straight line passes through the optimum point (15, 30) in the shaded region.
Therefore, the maximum number of pupils is
$= 4(15) + 8(30)$
$= 300$



4 . Check and reflect

- Take any point in the shaded region, for example (20, 20).
- Substitute this point (20, 20) into the function k.
  $k = 4(20) + 8(20)$
  $k = 240\ (< 300)$

CHAPTER 7

Self-Exercise 7.2

1. An institution offers two business courses, namely Management and Finance Courses. The number of students in the Management Course is x and the number of students in the Finance Course is y. The enrolment of these students is based on the following constraints.

I The total number of students in the Management and Finance Courses does not exceed 80 people.

II The number of students in the Finance Course does not exceed four times the number of students in the Management Course.

III The number of Finance Course students must exceed the number of Management Course students by at least 10 people.

(a) Other than $x \geq 0$ and $y \geq 0$, write three linear inequalities that satisfy all constraints above.

(b) Using a scale of 2 cm to 10 students on both axes, construct and label the region R that satisfies all of the above constraints.

(c) By using the graph in (b), find

(i) the range for the number of students in the Finance Course if the number of students in the Management Course is 20 people,

(ii) the maximum total of weekly fees that can be collected if the fees per week from the Management and Finance Courses students are RM60 and RM70 respectively.

2. A factory produces vases A and B using machines P and Q. The table below shows the time taken to produce each type of vase.

Vase	Time taken (minutes)	
	Machine P	Machine Q
A	40	30
B	20	60



Vase A



Vase B

The factory produces x units of vase A and y units of vase B a week. Machine P operates not more than 2 000 minutes while machine Q operates at least 1 800 minutes. The production of vase B does not exceed three times that of the production of vase A.

(a) Other than $x \geq 0$ and $y \geq 0$, write three inequalities that satisfy all constraints above.

(b) Using a scale of 2 cm to 10 units on both axes, construct and label the region R that satisfies all constraints.

(c) By using the graph from (b), find

(i) the minimum number of vase B produced if the factory intends to produce only 30 units of vase A,

(ii) the maximum profit per week if the profits from one unit of vase A and one unit of vase B are RM300 and RM250 respectively.

KEMENTERIAN PENDIDIKAN MALAYSIA

Formative Exercise 7.2

Quiz bit.ly/3lCsmia

1. A gardener wants to plant hibiscus and roses on his plot of land of 300 m^2. He has at least RM1 000 to buy the plants. A hibiscus plant costs RM4 and it requires a land area of 0.4 m^2 while a rose plant costs RM5 and it requires a land area of 0.3 m^2. The number of roses must exceed the number of hibiscus by at most 200.

(a) Other than $x \geq 0$ and $y \geq 0$, write three inequalities that satisfy all of the above constraints, if x represents the number of hibiscus plants and y represents the number of rose plants.

(b) Using a scale of 2 cm to 100 trees on the x-axis and the y-axis, draw and label the region R that satisfies all the inequalities in (a).

(c) From the graph obtained in (b), answer each of the following questions.

(i) Find the maximum number of rose plants if the number of hibiscus plants is 300.

(ii) Within a given period, each hibiscus and rose plant generates a profit of RM3.50 and RM2.40 respectively. Find the maximum profit of the gardener.

2. Mr Malik allocates RM3 000 to purchase x copies of science reference books and y copies of mathematics reference books for the school library. The average costs per copy of science reference books and mathematics reference books are RM30 and RM25 respectively. The number of science reference books purchased is at least 20 copies and the number of mathematics reference books purchased is at least 10 copies more than the science reference books.

(a) Write down three linear inequalities that satisfy all the given conditions other than $x \geq 0$ and $y \geq 0$.

(b) Using a scale of 2 cm to 20 copies of books on both axes, construct and label the region R that satisfies all the conditions.

(c) From the graph obtained in (b), find the minimum cost to purchase the books.

3. A beverage factory produces two types of beverages, P and Q. To meet consumers' demand, the factory must produce x litres of beverage P and y litres of beverage Q. The production of beverages from the factory is based on the following constraints.

I The total volume of beverages produced is not more than 7 000 litres.

II The volume of beverage Q produced is not more than twice the volume of beverage P produced.

III The volume of beverage Q produced is at least 1 000 litres.

(a) Write three linear inequalities, other than $x \geq 0$ and $y \geq 0$, which satisfy all the constraints above.

(b) Using a scale of 1 cm to 1 000 litres on the x-axis and the y-axis, construct and label the region R that satisfies all the above constraints.

(c) Based on the graph obtained in (b), answer each of the following questions.

(i) On a given day, the volume of beverage Q produced is 2 000 litres. Find the maximum volume of beverage P produced.

(ii) If the profits per litre of beverage P and Q are RM50 and RM30 respectively, find the maximum total profit of the factory.

KEMENTERIAN PENDIDIKAN MALAYSIA

REFLECTION CORNER

LINEAR PROGRAMMING

Given a straight line $ax + by = c$, where $b > 0$:

- Region above the straight line satisfies the inequalities $ax + by \geqslant c$ and $ax + by > c$.
- Region below the straight line satisfies the inequalities $ax + by \leqslant c$ and $ax + by < c$.

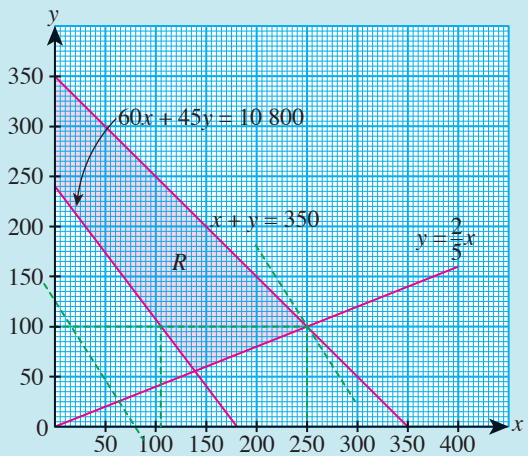
Applications

Steps to solve a linear programming problem:

1. Represent all the constraints for the situation in linear inequalities.
2. Draw a graph for each linear inequality and shade the feasible region.
3. Define the objective function $ax + by = k$ and draw a graph for that objective function.
4. Determine the optimal value (maximum or minimum value) by substituting the coordinates of the maximum point or the minimum point into the objective function.

Journal Writing

The diagram on the right shows the solution to determine the maximum profit of a business venture. R is a region that satisfies all the constraints in the business venture. Write a journal related to this business venture and present your findings to the class.



KEMENTERIAN PENDIDIKAN MALAYSIA

Summative Exercise

1. A family in a village produces two types of rattan chairs, namely small rattan chairs and big rattan chairs. The family is able to get at least 60 kg of rattan a week as the raw material. A small rattan chair requires 3 kg of rattan while a big rattan chair requires 5 kg of rattan. There are 60 workers. Two workers are required to produce one small rattan chair while three workers are required to produce one big rattan chair. PL 4

(a) If x number of small rattan chairs and y number of big rattan chairs are produced in a week, write four linear inequalities that satisfy the above conditions.

(b) Using a scale of 2 cm to 5 rattan chairs on both axes, construct and label the region R that satisfies all the linear inequalities.

(c) The price for a small rattan chair is RM40 and the price for a big rattan chair is RM80. From the graph obtained in (b), find

(i) the values of x and y that will provide the family with a maximum income,

(ii) the maximum income.

2. A baker takes 2.5 hours to bake an orange cake and 3 hours to bake a strawberry cake. The costs of making an orange cake and a strawberry cake are RM15 and RM20 respectively. In a week, the baker can bake x orange cakes and y strawberry cakes based on the following conditions. PL 5

I The baker works at least 30 hours a week.

II The total cost of baking both cakes is not more than RM300 a week.

III The number of orange cakes is not more than twice the number of strawberry cakes.

(a) Write three linear inequalities, other than $x \geq 0$ and $y \geq 0$, that satisfy all the constraints above.

(b) Using a 2 cm scale to represent 2 cakes on both axes, construct and label the region R that satisfies all the above constraints.

(c) Using the graph obtained in (b), find the maximum profit of the baker in a week if the profits from an orange cake and a strawberry cake are RM17 and RM20 respectively.

3. A post office wants to send 600 parcels to city M using x lorries and y vans. The transportation for the parcels are subjected to the following constraints. PL 5

I A lorry can carry 120 parcels while a van can carry 50 parcels.

II The number of vans used is not more than three times the number of lorries used.

III The number of vans used is at least 2.

(a) Other than $x \geq 0$ and $y \geq 0$, write three linear inequalities that satisfy all the constraints above.

(b) Using a 2 cm scale to a lorry on the x-axis and 2 cm to two vans on the y-axis, construct and label the region R that satisfies all the above constraints.

(c) Using the graph obtained in (b), find

(i) the range of the number of lorries if 2 vans are used,

(ii) the maximum cost incurred if the costs of transportation by a lorry and a van are RM150 and RM100 respectively.

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4. Setia Indah Secondary School will host a motivational camp. Participants of the camp are made up of x female pupils and y male pupils. The fee for a female pupil is RM100 and the fee for a male pupil is RM120. The number of pupils in the camp is based on the following constraints. PL 5

I The maximum number of pupils attending the camp is 80.

II The ratio of the number of female pupils to male pupils is at least 1 : 3.

III The total fees collected is not less than RM5 000.

(a) Write three linear inequalities that satisfy all the above constraints other than $x \geq 0$ and $y \geq 0$.

(b) Using a 2 cm scale for 10 pupils on the x-axis and the y-axis, construct and label the region R that satisfies all the above constraints.

(c) Using the graph obtained in (b), find

(i) the minimum number of male pupils if the ratio of the number of female to male pupils is 1 : 3,

(ii) the maximum profit obtained if the school takes 25% of the total fees collected.

5. A factory produces two types of cupboards, namely cupboard A and cupboard B. Each cupboard requires two types of raw materials P and Q. The amount of each raw material needed to produce each unit of cupboard A and cupboard B are shown in the table below. PL 6

Cupboard	Number of raw material	
	P	Q
A	2	3
B	5	2



Cupboard A



Cupboard B

The amount of raw materials P and Q available to the factory are 30 units and 24 units respectively. It is given that the number of cupboard A produced is at most twice the number of cupboard B. Suppose the factory produces x units of cupboard A and y units of cupboard B.

(a) Write three linear inequalities, other than $x \geq 0$ and $y \geq 0$, which satisfy all the constraints above.

(b) Using a scale of 2 cm to 2 units on the x-axis and 2 cm to 1 unit on the y-axis, construct and label the region R that satisfies all the above constraints.

(c) Based on the graph obtained in (b), find

(i) the maximum number of cupboard B produced if the factory produces 4 units of cupboard A,

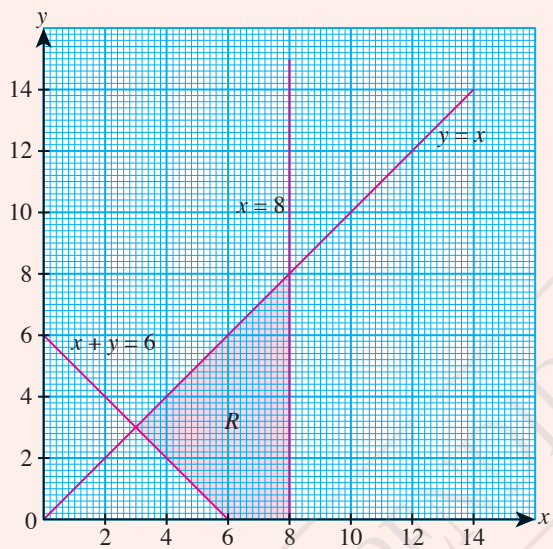
(ii) the maximum profit earned by the factory if the profit from one unit of cupboard A is RM200 and one unit of cupboard B is RM250.

KEMENTERIAN PENDIDIKAN MALAYSIA

MATHEMATICAL EXPLORATION

(a) In your group, discuss the following situation using Hot Seat activity. It is given that the region on one side of a straight line $ax + by = c$. If $b < 0$, which region satisfies $ax + by \geqslant 0$?

(b) A school is given an allocation to purchase type A computers and type B computers for its computer lab. The purchase of the computers is based on the conditions represented on the region R in the diagram below. The total number of computers purchased is at least 6 units.



(i) State what are represented by the x-axis and the y-axis.

(ii) Besides the numbers of type A computers or type B computers being greater than zero, write three other conditions in sentences.

(iii) If the school purchased 6 units of type A computers, what is the maximum number of type B computers that can be bought?

(iv) If the costs of one type A computer and one type B computer are RM1 500 and RM2 000 respectively, find the maximum allocation required by that school.

Information Corner

Learning steps of Hot Seat activity.

1. An expert pupil will sit on a chair.
2. Pupils in groups will ask questions related to the problem.
3. The expert pupil will answer all the questions.
4. Each group will make conclusions for all the problems.

KEMENTERIAN PENDIDIKAN MALAYSIA

CHAPTER 8

KINEMATICS OF LINEAR MOTION



What will be learnt?

- Displacement, Velocity and Acceleration as a Function of Time
- Differentiation in Kinematics of Linear Motion
- Integration in Kinematics of Linear Motion
- Applications of Kinematics of Linear Motion

List of Learning Standards

bit.ly/2EThNrb

A drone, unmanned aerial vehicle equipped with a camera, is a modern technology tool to assist humans with their tasks. For example, drones are used in transportation, agricultural sectors, surveying and mapping. Drones can fly to an altitude of 500 m and still be able to take good quality pictures. In your opinion, what is the maximum distance a drone can fly? At what velocity does the drone have to fly in order to take high quality pictures?

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Info Corner

Kinematics is a study of the movement of an object without reference to the forces that cause its movement.

A scalar quantity refers to a quantity with magnitude while a vector quantity refers to a quantity that has magnitude and direction.

For more info:

bit.ly/37eXwVs

Significance of the Chapter

- Knowledge of kinematics is important to solve problems in the fields of engineering, robotics, biomechanics, sports science and astronomical science.
- Knowledge of kinematics assists us with problems associated with time, velocity and acceleration.

Keywords

Displacement	*Sesaran*
Velocity	*Halaju*
Acceleration	*Pecutan*
Distance	*Jarak*
Initial velocity	*Halaju awal*
Uniform velocity	*Halaju malar*
Maximum velocity	*Halaju maksimum*
Minimum velocity	*Halaju minimum*
Uniform acceleration	*Pecutan malar*
Positive velocity	*Halaju positif*
Negative velocity	*Halaju negatif*
Zero velocity	*Halaju sifar*
Positive displacement	*Sesaran positif*
Negative displacement	*Sesaran negatif*
Zero displacement	*Sesaran sifar*

Video about the motion of drone

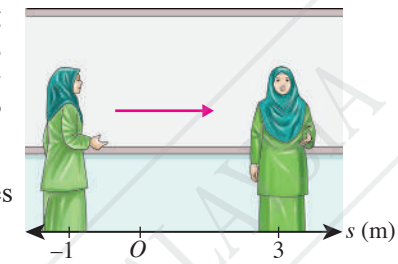
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KEMENTERIAN PENDIDIKAN MALAYSIA

8.1 Displacement, Velocity and Acceleration as a Function of Time

Describing and determining instantaneous displacement, instantaneous velocity and instantaneous acceleration of a particle

The diagram shows the initial position of a teacher standing 1 metre on the left of a fixed point O. Then, the teacher moves to the position of 3 metre to the right of O. What can you say about the position of the teacher in relation to the fixed point O?



If O is a reference point and the teacher is standing 3 metres to the right of O, her displacement is positive 3 metres from O, which is $s = 3$ m. When the teacher is 1 metre to the left of O, her displacement is negative 1 metre from O, which is $s = -1$ m. When she is at point O, her displacement is zero metre, which is $s = 0$ m.

Displacement, s of a particle from a fixed point is the distance of the particle from the fixed point measured on a certain direction.

Flash Quiz

Besides displacement, give three examples of physical quantities that are vector quantities.

A displacement is a vector quantity that has a magnitude and a direction. Hence, the value of displacement can be positive, zero or negative. On the other hand, **distance** is a scalar quantity that refers to the total path travelled by an object.

Follow the exploration below to find out more about instantaneous displacement and the position of a particle during its movement.

Discovery Activity 1 Group 21st cl

Aim: Describe and determine an instantaneous displacement and the position of a particle

Steps:

1. Read and understand the following situations.

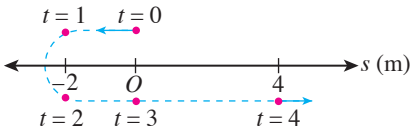
   A particle moves along a straight line from a fixed point O. The displacement of the particle, s m, from O at t seconds after passing O is given by $s = t^2 - 3t$.

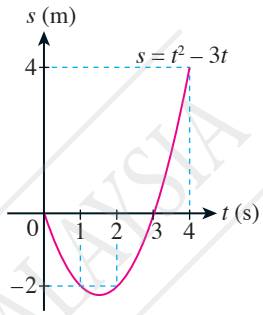
2. Copy and complete the table below for $s = t^2 - 3t$ for $0 \leqslant t \leqslant 4$.

Time, t (s)	0	1	2	3	4
Displacement, s (m)					

3. What can you say about the displacement of the particle when $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 4$?
4. If the movement of the particle to the right is considered positive, construct a number line to represent the positions of the particle and sketch a displacement-time graph.
5. State the position of the particle relative from the point O when the displacement is
   (a) negative, (b) zero, (c) positive.
4. Discuss your findings with group members and present your findings to the class.

From Discovery Activity 1, the value of displacements obtained represent the displacements of the particle at time $t = 0$, $t = 1$, $t = 2$, $t = 3$ and $t = 4$. The displacement of a particle at a certain time is called **instantaneous displacement**. The position of the particle can be illustrated using a number line and a displacement-time graph as shown.





From the number line and the displacement-time graph:

- The displacement is negative for $0 < t < 3$ and the particle is on the left of fixed point O or below the t-axis in this interval.
- The displacement is zero at $t = 0$ and $t = 3$. The particle is at the fixed point O or on the t-axis.
- The displacement is positive for $t > 3$ and the particle is on the right of fixed point O or above of the t-axis in this interval.

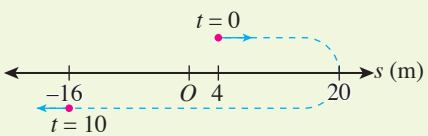
In general,

If O is a fixed point and the movement of a particle to the right is positive, then
- The displacement is negative, $s < 0$, meaning the particle is on the left of point O.
- The displacement is zero, $s = 0$, meaning the particle is on the point O.
- The displacement is positive, $s > 0$, meaning the particle is on the right of point O.

Example 1

A particle moves along a straight line and passes through a fixed point O. The displacement, s metre, of the particle t seconds after it starts moving is given as $s = 4 + 8t - t^2$. Calculate the instantaneous displacement, in m, and determine the position of the particle from point O when
(a) $t = 0$ (b) $t = 10$

Solution



Given $s = 4 + 8t - t^2$.

(a) When $t = 0$, $s = 4 + 8(0) - (0)^2$
$s = 4$
Therefore, the particle is located 4 m to the left from the fixed point O when $t = 0$.

(b) When $t = 5$, $s = 4 + 8(10) - (10)^2$
$s = 4 + 80 - 100$
$s = -16$
Therefore, the particle is located 16 m from the fixed point O when $t = 10$.

Example 2

A particle moves along a straight line and passes through a fixed point O. Its displacement, s m, t seconds after passing point O is given by $s = 4t - t^2$ for $0 \leqslant t \leqslant 5$. Represent the displacement of the particle by using
(a) the number line, (b) the displacement-time graph.

CHAPTER 8

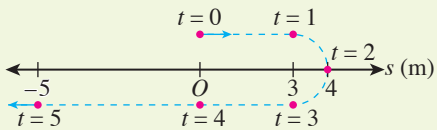
KEMENTERIAN PENDIDIKAN MALAYSIA

Solution

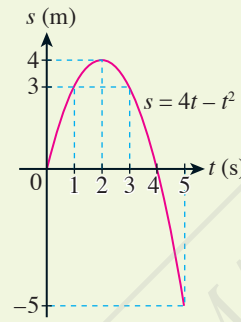
Given $s = 4t - t^2$. Construct a table for the displacement of particle, $s = 4t - t^2$ for $0 \leqslant t \leqslant 5$.

Time, t (s)	0	1	2	3	4	5
Displacement, s (m)	0	3	4	3	0	–5

(a)



(b)



Consider a race car that can reach a speed of over 350 kmh^{-1}. The movement of this race car involves speed and velocity.

A **velocity**, v is the rate of change of displacement with time while **speed** is defined as the rate of change of distance with time. Since a velocity has a magnitude and a direction, then it is the vector quantity while a speed is a scalar quantity for the movement of a particle.

Let's explore ways to determine the instantaneous velocity and its direction at a given time, t of a pupil's run.

Discovery Activity 2

Group 21st cl

Aim: Describe and determine the instantaneous velocity and the direction of a pupil.

Steps:

1. Examine the situation below.

   A pupil is running along a straight track from a fixed point O. His displacement, s m, t seconds after passing through O is given by $s = 8t - 2t^2$. The displacements of the pupil are recorded at $t = 0$ until $t = 6$.

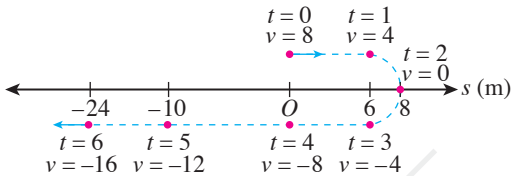
2. Assume the movement to the right is positive, represent the displacement of the pupil's run using
   (a) a number line, (b) a displacement-time graph.
3. From the displacement-time graph obtained, find the gradients of the tangent to the graph at $t = 0, 1, 2, 3, 4, 5$ and 6.
4. By using the relationship $v = 8 - 4t$, such that v is the velocity and t is the time, determine the values of v by substituting the values of t obtained in Step 3 in the velocity function v and also pay attention to the positive and negative values.

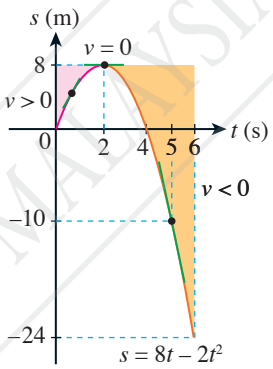
From Discovery Activity 2, the number line and the gradient of tangent at one point on the displacement-time graph can be used to determine the velocity and the direction of the pupil. It is found that the gradient of tangent at a certain time is the same as the pupil's velocity at that time. For instance, when $t = 5$, the gradient of tangent is –12, therefore the velocity of the pupil is $-12\ ms^{-1}$. The velocity of an object at a certain time is called an **instantaneous velocity**.

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From the number line and the displacement-time graph:

- The gradient of tangent for $0 \leqslant t < 2$ is positive, so the velocity of the pupil is positive, which is $v > 0$. The pupil is running to the right from O.
- At $t = 2$, the gradient of tangent is zero, so the velocity of the pupil is zero, which is $v = 0$. The pupil is instantaneously at rest before changing his movement.
- The gradient of tangent for $t > 2$ is negative, so the velocity of the pupil is negative, which is $v < 0$. The pupil is running to the left through O in this duration.





In general,

If O is a fixed point and the movement of a particle to the right is positive, then

- The velocity is positive, $v > 0$, meaning that the particle moves to the right.
- The velocity is zero, $v = 0$, meaning that the particle is at rest, that is, the particle is stationary.
- The velocity is negative, $v < 0$, meaning that the particle moves to the left.

Example 3

A particle moves along a straight line and passes through a fixed point O. Its velocity, v ms^{-1}, t seconds after passing through the point O is given by $v = 3t - 12$.

(a) Calculate
  (i) the initial velocity of the particle,
  (ii) the instantaneous velocity, in ms^{-1}, of the particle when $t = 5$,
  (iii) the time, in seconds, when the instantaneous velocity, in ms^{-1}, of the particle is 6 ms^{-1}.

(b) Sketch the velocity-time graph to represent the movement of the particle for $0 \leqslant t \leqslant 6$.

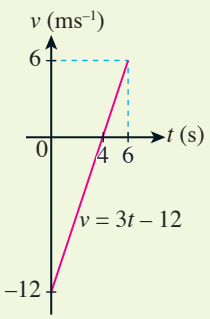
Solution

(a) (i) When $t = 0$, $v = 3(0) - 12$
$v = -12$
Hence, the initial velocity of the particle is –12 ms^{-1}.

(ii) When $t = 5$, $v = 3(5) - 12$
$v = 15 - 12$
$v = 3$
Hence, the instantaneous velocity of the particle when $t = 5$ is 3 ms^{-1}.

(iii) $3t - 12 = 6$
$3t = 18$
$t = 6$
Hence, the time is 6 seconds when the instantaneous velocity of the particle is 6 ms^{-1}.

(b)



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Acceleration, a is the rate of change of velocity with time. Then, the acceleration function, a is a function of time, $a = f(t)$ and is a vector quantity that has magnitude and direction.

If the rate of change of velocity with time of an object that moves is the same at any time, then the object is moving with a **uniform acceleration**. On the other hand, if the rate of change of velocity with time is different at any time, the object is moving with a **non-uniform acceleration**.

Information Corner

When the velocity of an object decreases, the object decelerates and the value of the acceleration becomes negative.

An acceleration, a at a certain time, t is called an **instantaneous acceleration** and can be obtained by determining the gradient of tangent of velocity-time graph at time, t.

Let's explore the following activity to determine the instantaneous acceleration of a woman swimming along a straight path.

Discovery Activity 3

Group 21st cl

Aim: To describe and determine the instantaneous acceleration of a swimmer

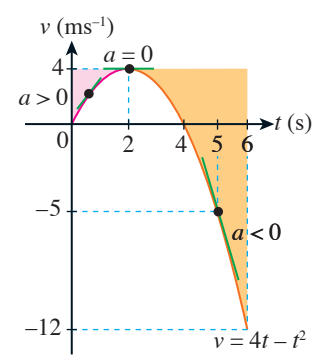
Steps:

1. Form a few groups. Then, examine the situation below.

   A woman swims along a straight path. Her velocity, v ms^{-1}, at t seconds from the initial point O is given by $v = 4t - t^2$. The velocity of the swimmer is recorded at time $t = 1$, $t = 2$, $t = 3$, $t = 4$ and $t = 5$.

2. Each group needs to answer each of the following questions.
   (a) Represent the movement of the swimmer using a velocity-time graph.
   (b) Find the gradient of the tangent to the curve when $t = 1$, $t = 2$, $t = 3$, $t = 4$ and $t = 5$.
   (c) What can you say about the acceleration of the swimmer when $t = 1$, $t = 2$, $t = 3$, $t = 4$ and $t = 5$?
   (d) Draw a conclusion when
   (i) $a > 0$ (ii) $a = 0$ (iii) $a < 0$
3. Discuss the findings in groups.
4. Appoint a representative in your group to present the results to the class.

From Discovery Activity 3, the gradient of tangent at one point of the velocity-time graph can be used to determine the acceleration of the swimmer. For instance, when $t = 5$, the gradient of tangent is -6, so the acceleration of the swimmer when $t = 5$ is -6 ms^{-1}. An acceleration of an object at a certain time is called an **instantaneous acceleration**.



How do you describe the movement of an object when its instantaneous acceleration is negative? What is the difference in the motion of an object when it has the instantaneous acceleration of -6 ms^{-2} and 6 ms^{-2}? Explain.

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From the velocity-time graph in page 256:

- For the time interval $0 \leqslant t < 2$, the gradient of tangent is positive, which is $a > 0$ and v is increasing. Then, the acceleration of the swimmer is positive for this time interval and the swimmer is accelerating.
- At $t = 2$, the gradient of tangent is zero, which is $a = 0$ and the velocity, v is maximum. Then, the acceleration of the swimmer at this time is zero. Zero acceleration does not mean the velocity is zero but rather it is either maximum or minimum.
- For $t > 2$, the gradient of tangent is negative, which is $a < 0$ and v is decreasing. Then, the acceleration of the swimmer is negative for this time interval and the swimmer is decelerating.

In general,

If the movement of a particle to the right is positive, then
- The acceleration is positive, $a > 0$, meaning the velocity of particle is increasing with time.
- The acceleration is zero, $a = 0$, meaning the velocity of particle is either maximum or minimum.
- The acceleration is negative, $a < 0$, meaning the velocity of particle is decreasing with time.

Example 4

A particle moves along a straight line and passes through a fixed point O. At t seconds after passing through O, its acceleration, a ms^{-2}, is given by $a = 12 - 4t$. Calculate the instantaneous acceleration, in ms^{-2}, of the particle when $t = 7$.

Solution

Given $a = 12 - 4t$.
When $t = 7$, $a = 12 - 4(7) = -16$
Therefore, the instantaneous acceleration of the particle when $t = 7$ is -16 ms^{-2}.

Information Corner

The negative value of the acceleration shows that the particle is slowing down or decelerating.

Self-Exercise 8.1

1. A particle moves along a straight line and passes through a fixed point O. Its displacement, s m, is given by $s = 2t^2 - 5t - 3$, where t is the time in seconds after the movement begins.
(a) Find the instantaneous displacement, in m, of the particle when
(i) $t = 0$ (ii) $t = 2$
(b) Find the time when the particle
(i) passes through point O, (ii) is 9 m to the right of point O.
(c) Determine the range of time, in seconds, when the particle is to the right of point O.

2. A particle moves along a straight line and passes through a fixed point O. Its velocity, v ms^{-1}, is given by $v = t^2 - 8t + 7$, where t is the time in seconds after passing through O.
(a) Find the instantaneous velocity, in ms^{-1}, of the particle when $t = 3$.
(b) Calculate the values of t, in seconds, when the particle stops instantaneously.
(c) Determine the range of values of t, in seconds, when the particle moves to the left.

3. A particle moves along a straight line and passes through a fixed point O. Its acceleration, a ms^{-2}, is given by $a = 8 - 4t$, where t is the time in seconds after passing through O.
(a) Find the instantaneous acceleration, in ms^{-2}, of the particle when $t = 4$.
(b) Calculate the time, in seconds, when the velocity of the particle is maximum.
(c) Determine the range of time, in seconds, when the velocity of the particle is increasing.

Determining the total distance travelled by a particle within a given period of time

The displacement of a particle can be observed by drawing a number line or sketching a displacement function graph, $s = f(t)$. From the number line and the graph, the total distance travelled by the particle in a given period of time can be determined easily.

Example 5

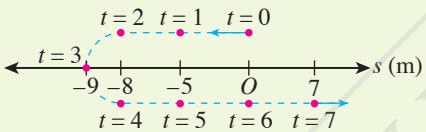
A particle moves along a straight line from a fixed point O. Its displacement, s m, t seconds after passing through O is given by $s = t^2 - 6t$. Find the total distance, in m, travelled by the particle in the first 7 seconds.

Solution

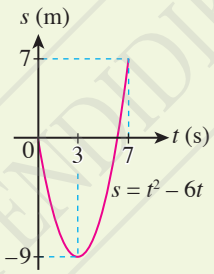
Given $s = t^2 - 6t$.

Time, t (s)	0	1	2	3	4	5	6	7
Displacement, s (m)	0	–5	–8	–9	–8	–5	0	7

Number line:



Displacement-time graph:



DISCUSSION

Based on Example 5, is the distance travelled in the first 7 seconds the same as the displacement at 7th second? How about the distance travelled in the 7th second? Discuss.

The total distance travelled by the particle in the first 7 seconds
$= 9 + 9 + 7$
$= 25$ m

Self-Exercise 8.2

1. A particle moves along a straight line from a fixed point O. Its displacement, s m, t seconds after passing through O is given by $s = 4t^2 + t$. Calculate the total distance, in m, travelled by the particle
(a) when $0 \leqslant t \leqslant 4$,
(b) from $t = 3$ to $t = 6$.

2. A particle moves along a straight line and passes through a fixed point O. Its displacement, s m, t seconds after it starts moving is given by $s = 6t - t^2 + 7$. The particle moves to the right of O until $t = 3$ and then moves towards O again. Find
(a) the total distance, in m, travelled by the particle
(i) in the first 2 seconds,
(ii) in the first 9 seconds.
(b) the distance, in m, travelled by the particle at the 7th second.

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Formative Exercise 8.1

Quiz bit.ly/3iNq6Tr

1. The mud deposited on the river bed makes the river that runs through a village shallow. This makes the boat travelling in and out difficult. A boat that moves along a jetty in a straight line with displacement, s metre, t seconds after passing the jetty is given by $s = t^2 - 4t$.

(a) Copy and complete the table below.

Time, t (seconds)	1	2	3	4	5
Displacement, s (metre)					

(b) Sketch the displacement-time graph to show the positions of the boat.

(c) Find the time, in seconds, when the boat is at the jetty again.

2. Syaza cycles her three-wheel bicycle in a straight line at her house backyard and has an initial displacement of 2 metres from a flower pot. Her displacement, s metre, t seconds after passing the flower pot is given by $s = t^3 + 2t + c$.

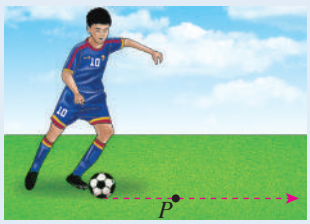
(a) Determine the value of c.

(b) Find the distance of Syaza from the flower pot when

(i) $t = 2$ (ii) $t = 3$

3. A particle moves along a straight line from a fixed point O. Its displacement, s m, t seconds after passing through O is given by $s = 3t^2 + 2t$. Calculate the instantaneous displacement of the particle when $t = 0$ and $t = 10$.

4. The diagram on the right shows a boy kicking a ball in a field. The ball moves in a straight line and passes through a fixed point marked P. At t seconds after passing point P, the velocity, v ms^{-1}, of the ball is given by $v = 7t - 5$. Find the instantaneous velocity, in ms^{-1}, of the ball when $t = 2$ and $t = 4$.



5. A particle moves along a straight line and passes through a fixed point O. Its acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 4 - 2t$.

(a) Find the initial acceleration, in ms^{-2} of the particle.

(b) Determine the range of time, in seconds, when the velocity of the particle is decreasing.

6. A particle moves along a straight line and passes through a fixed point O. Its displacement, s m, at t seconds after passing through point O is given by $s = 2t^2 + t$. Calculate

(a) the displacement, in m, of the particle when $t = 3$,

(b) the total distance, in m, travelled by the particle in the first 5 seconds.

7. A particle moves along a straight line. At time t seconds after it starts moving, its displacement, s m, from the fixed point O is given by $s = (t - 2)^2 + 5$.

(a) Copy and complete the table below.

Time, t (seconds)	0	1	2	3	4	5	6
Displacement, s (metre)							

(b) Sketch the displacement-time graph for $0 \leqslant t \leqslant 6$.

(c) Calculate the total distance, in m, travelled by the particle in the first 6 seconds.

8.2 Differentiation in Kinematics of Linear Motion

Relationship between displacement function, velocity function and acceleration function

In differentiation, for a function $y = f(x)$, its derivative $\frac{dy}{dx}$ can be considered as the rate of change of y with respect to x. This concept can be used for the movement of a particle along a straight line. For instance, displacement, s of a moving particle is a function of time, t which is $s = f(t)$. So, its derivative $\frac{ds}{dt}$ is a rate of change of s with respect to t.

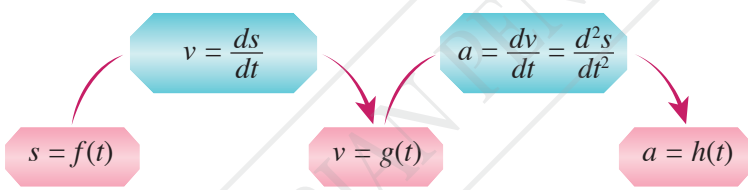
Thus, the velocity function of particle at time t, $v = g(t)$ is given by:

$$v = \frac{ds}{dt}$$

An acceleration, a is a rate of change of velocity with time and its function, $a = h(t)$ is given by:

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

The relationship between the displacement function, $s = f(t)$, velocity function, $v = g(t)$ and acceleration function, $a = h(t)$ can be simplified as follows:



HISTORY GALLERY



Isaac Newton is the first person to introduce calculus and differentiation. His book entitled Philosophiae Naturalis Principia Mathematica became the foundation to the idea of limit in differentiation.

Recall

If $y = ax^n$, then $\frac{dy}{dx} = anx^{n-1}$, where a is an integer and n is a constant.

Example 6

A particle moves along a straight line. At time t seconds after it starts moving, its displacement, s m, from the fixed point O is given by $s = 3 + 2t - t^2$, where t is time, in seconds.

(a) Determine the velocity function, v and acceleration function, a of the particle.

(b) On the same diagram, sketch a graph of functions s, v and a for $0 \leqslant t \leqslant 3$ and explain the motion of the particle from point O for that interval.

Solution

(a) Given the displacement function, $s = 3 + 2t - t^2$

Then, the velocity function at time t, $v = \frac{ds}{dt}$

$$v = 2 - 2t$$

and the acceleration function at time t, $a = \frac{dv}{dt}$

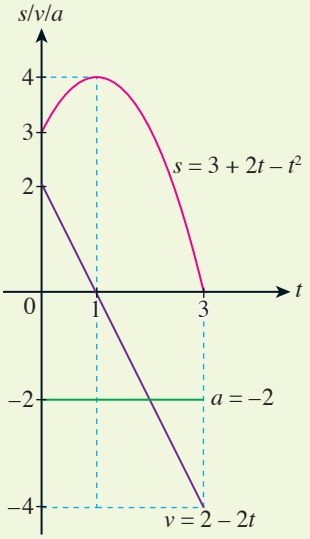
$$a = -2$$

Excellent Tip

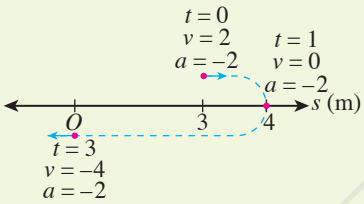
$a = -2$ means the particle is moving with a uniform acceleration of -2 ms^{-2}.

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(b)



Graph of displacement, velocity and acceleration functions of the particle that moves from the fixed point O can be simplified on a number line as follows:



From the graphs and the number line:

- It is found that the displacement of the particle at $t = 0$ from the fixed point O is 3 m, the initial velocity is 2 ms^{-1} and the acceleration is -2 ms^{-2}.
- At $t = 1$, the particle changes its direction, the displacement from the fixed point O is maximum, which is 4 m, the velocity is 0 ms^{-1} and the acceleration is -2 ms^{-2}.
- At $t = 3$, the particle reaches to the fixed point O where its displacement is 0 m, velocity is 2 ms^{-1} and its acceleration is the same, which is -2 ms^{-2}.
- The total distance travelled by the particle from $t = 0$ until $t = 3$ is $(4 - 3) + 4 = 5$ m.

Self-Exercise 8.3

1. Determine the velocity function, v in terms of t for a particle that moves along a straight line in each of the following using differentiation.

(a) $s = t(2 - t)^2$
(b) $s = 16t - t^2$
(c) $s = 2t^3 - 4t^2 + 2t + 1$
(d) $s = t^3(3 + t)^2$
(e) $s = t(2t^2 - 9t - 5)$
(f) $s = \frac{1}{3}t^3 - 3t^2 + 5t - 2$

2. Determine the acceleration function, a in terms of t of a particle that moves along a straight line for each of the following.

(a) $s = \frac{1}{3}t^3 - \frac{1}{2}t^2 + 4t$
(b) $s = t^3 - 5t^2 + 7$
(c) $s = 8t - 2t^3$
(d) $v = (5 - 3t)^2$
(e) $v = 3t^2 - \frac{1}{t} + 4$
(f) $v = 6t^3 - \frac{4}{t^2}$

3. A particle moves along a straight line and passes through a fixed point O. Its displacement, s m, is given by $s = 8 + 2t - t^2$, where t is the time in seconds after it starts moving.

(a) Determine the expressions for the velocity function, v and acceleration function, a of the particle in terms of t.
(b) Sketch graphs of displacement, velocity and acceleration functions of the particle for $0 \leqslant t \leqslant 4$. Then, interpret the graphs.

CHAPTER 8

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Determining and interpreting instantaneous velocity of a particle from acceleration function

We have learnt that velocity is the rate of change of displacement with respect to time. Thus, if given displacement function, $s = f(t)$, the velocity function v at time t can be determined by differentiating s with respect to t, which is $v = \frac{ds}{dt}$. From the velocity function obtained, can you determine the instantaneous displacement of a particle at any time? Let's explore the following activity.

Discovery Activity 4

Group | 21st cl | STEM | CT

Aim: To determine and interpret the instantaneous velocity of a particle from the displacement function

Steps:

1. Examine the situation below.

   A particle moves along a straight line. Its displacement, s metre from a fixed point O at t seconds is represented by the displacement function, $s = 40t - 5t^2$, such that $0 \leqslant t \leqslant 10$.

2. Scan the QR code on the right or visit the link below it to see the motion of particle on a displacement-time graph for function $s = 40t - 5t^2$ for $0 \leqslant t \leqslant 10$.

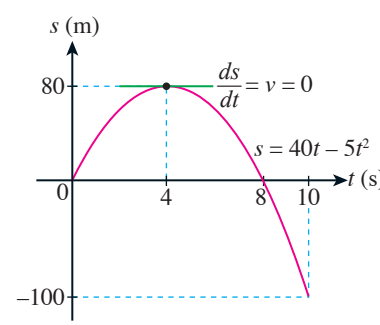
   ggbm.at/z7r9kbtc

3. Drag the point A along the curve of graph to see the gradient of tangent at point A to the graph.
4. What can you say about the gradient of tangent to the curve when point A changes along the curve? Does the gradient change accordingly?
5. Copy and complete the table below to find the gradient of tangent, $\frac{ds}{dt}$ to the curve of graph at given time, t.

Time, t (s)	0	4	8	10
Gradient of tangent, $\frac{ds}{dt}$				

6. What can you say about the gradient of tangent, $\frac{ds}{dt}$ to the curve at time t obtained in the table above? Is the gradient of tangent, $\frac{ds}{dt}$ at time t obtained the instantaneous velocity of the particle at that time? Discuss.

From Discovery Activity 4, it is found that each of the gradient of the tangent, $\frac{ds}{dt}$ at $t = 0$, $t = 4$, $t = 8$ and $t = 10$ obtained is the instantaneous velocity from the displacement-time graph in the shape of curve, $s = 40t - 5t^2$ at time t.



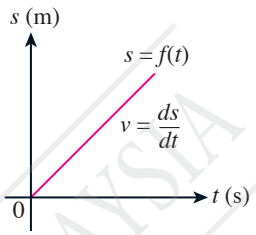
For the displacement-time graph in the shape of curve, the instantaneous velocity is different at every point on the curve.

For instance, at $t = 0$, its instantaneous velocity is 40 ms^{-1} and this velocity is called **initial velocity** of particle.

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At $t = 4$ where the displacement of particle is maximum, the instantaneous velocity is 0 ms^{-1}. The displacement of the particle in that time is called **maximum displacement**. The maximum or minimum displacement occurs when the gradient of tangent or instantaneous velocity of the particle is zero, that is $\frac{ds}{dt} = v = 0$.

For a linear displacement-time graph, the gradient of tangent at any point is the same. Thus, the instantaneous velocity of particle at any time is uniform. This velocity that is uniform is called **constant velocity**.



By differentiation, an instantaneous velocity of a particle at a certain time can be determined as follows:

Given the displacement function, $s = 40t - 5t^2$.

Then, the velocity function, $v = \frac{ds}{dt}$

$$v = 40 - 10t$$

When $t = 4$, the velocity, $v = 40 - 10(4)$

$$v = 0$$

Thus, the instantaneous velocity at 4 seconds is 0 ms^{-1}.

Excellent Tip

Maximum or minimum displacement occurs when $\frac{ds}{dt} = v = 0$.

In general,

An instantaneous velocity of a particle that moves along a straight line from a fixed point from a displacement function, $s = f(t)$ can be determined by substituting the value of t in the velocity function, $v = \frac{ds}{dt}$.

Example 7

A particle moves along a straight line so that its displacement, s metre after passing through a fixed point O is given by $s = t^3 - 9t^2 + 24t + 5$, where t is the time, in seconds, after movement started. Calculate

(a) the initial velocity, in ms^{-1}, of the particle,
(b) the instantaneous velocity, in ms^{-1}, at 3 seconds,
(c) the values of t, in seconds, when the particle is instantaneously at rest,
(d) the range of t, in seconds, when the particle moves to the left.

Solution

Given displacement function, $s = t^3 - 9t^2 + 24t + 5$, then the velocity function,
$v = \frac{ds}{dt} = 3t^2 - 18t + 24$

(a) When $t = 0$, $v = 3(0)^2 - 18(0) + 24$

$$v = 24$$

Hence, the initial velocity of the particle is 24 ms^{-1}.

(b) When $t = 3$, $v = 3(3)^2 - 18(3) + 24$

$$v = 27 - 54 + 24$$

$$v = -3$$

Hence, the instantaneous velocity of the particle at 3 seconds is –3 ms^{-1}.

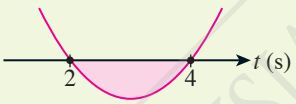
(c) When the particle rests for a while, $v = 0$

$$3t^2 - 18t + 24 = 0$$
$$t^2 - 6t + 8 = 0$$
$$(t - 2)(t - 4) = 0$$
$$t = 2 \text{ or } t = 4$$

Hence, the particle rests instantaneously at 2 seconds and 4 seconds.

(d) When the particle moves to the left, $v < 0$

$$3t^2 - 18t + 24 < 0$$
$$t^2 - 6t + 8 < 0$$
$$(t - 2)(t - 4) < 0$$



From the graph, the solution is $2 < t < 4$.

Hence, the particle moves to the left when $2 < t < 4$.

Self-Exercise 8.4

1. A particle moves along a straight line and passes through a fixed point O. Its displacement, s metres from O is given by $s = 2t^2 - 3t + 6$, where t is the time in seconds after the motion begins. Calculate

(a) the instantaneous velocity of the particle, in ms^{-1}, when

(i) $t = \frac{1}{4}$ (ii) $t = 2$ (iii) $t = 6$

(b) the time, in seconds, when the instantaneous velocity of the particle is

(i) $-1\ \text{ms}^{-1}$ (ii) $5\ \text{ms}^{-1}$ (iii) $9\ \text{ms}^{-1}$

2. A particle moves along a straight line. Its displacement, s metres from the fixed point O at time t is given by $s = 2t^3 - 5t^2 + 4t$. Find

(a) the instantaneous velocity, in ms^{-1}, of the particle when $t = 2$,

(b) the values of t, in seconds, when the particle stops instantaneously,

(c) the range of values of t, in seconds, when the particle moves to the right.

Determining and interpreting the acceleration of a particle from a velocity function and a displacement function

The gradient of the tangent to velocity function graph, $v = f(t)$ for the motion of a particle is the value of $\frac{dv}{dt}$ at time t, which is the instantaneous acceleration, a of the particle. The instantaneous acceleration, a of a particle moving in a straight line is also the rate of change of velocity with respect to time.

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{ds}{dt}\right) = \frac{d^2s}{dt^2}$$

On the velocity-time graph in Diagram 8.1, the gradient at any point on the graph is the same, that is, the rate of change of velocity with respect to time, $\frac{dv}{dt}$ at any moment is the same. Thus, the particle is said to have a uniform acceleration for its motion. This uniform acceleration is known as **constant acceleration**.

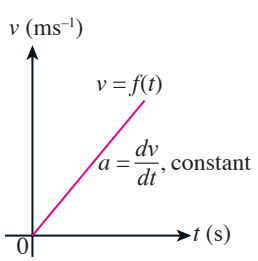


Diagram 8.1

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In Diagram 8.2, for the time interval $0 \leqslant t < a$, the velocity increases with time, so the instantaneous acceleration of the particle, $a = \frac{dv}{dt}$ at any point in this section is positive, which is $a > 0$.

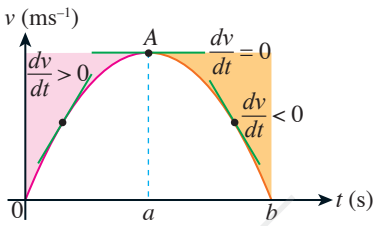


Diagram 8.2

On the other hand, for the time interval $a < t \leqslant b$, the velocity of the particle decreases with time, so the instantaneous acceleration of the particle, $a = \frac{dv}{dt}$ at any point in this part is negative, which is $a < 0$. This negative acceleration is known as **deceleration**.

At point A, the particle experiences a **maximum velocity** and its acceleration, $a = \frac{dv}{dt}$ at this point is zero, which is $a = 0$. Zero acceleration does not necessarily mean velocity is zero too. In fact, the velocity is either a maximum or a minimum.

In general,

Instantaneous acceleration, a of a particle moving along a straight line and passes through a fixed point can be determined from a velocity function, $v = f(t)$ or a displacement function, $s = f(t)$ by substituting the value of t into the acceleration function $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Example 10

A particle starts from a fixed point O and moves along a straight line. After t seconds, its displacement, s metre is given by $s = t^3 - 3t^2 - 4t$. Calculate
(a) the initial acceleration, in ms^{-2}, of the particle,
(b) instantaneous acceleration of the particle, in ms^{-2}, at 5 seconds,
(c) the acceleration of the particle, in ms^{-2}, when it passes through point O again,
(d) the range of values of t, in seconds, when the acceleration of the particle is positive.

Solution

Given the displacement function, $s = t^3 - 3t^2 - 4t$

Then, velocity function, $v = \frac{ds}{dt} = 3t^2 - 6t - 4$ and acceleration function, $a = \frac{dv}{dt} = 6t - 6$

(a) When $t = 0$, $a = 6(0) - 6 = -6$
Hence, the initial acceleration is -6 ms^{-2}.

(b) When $t = 5$, $a = 6(5) - 6 = 24$
Hence, the instantaneous acceleration at 5 seconds is 24 ms^{-2}.

(c) When the particle passes through O again,
$s = 0$
$t^3 - 3t^2 - 4t = 0$
$t(t^2 - 3t - 4) = 0$
$t(t + 1)(t - 4) = 0$
$t = 0, t = -1$ or $t = 4$
When $t = 4$, $a = 6(4) - 6 = 18$
Hence, when the particle passes through O again, its acceleration is 18 ms^{-2}.

(d) For the acceleration to be positive, $a > 0$
$6t - 6 > 0$
$6t > 6$
$t > 1$
Hence, the acceleration of the particle is positive when $t > 1$.

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Self-Exercise 8.5

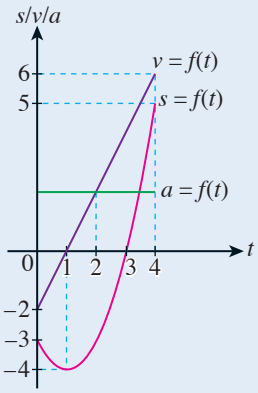
1. A particle moves along a straight line. Its velocity, v ms^{-1}, t seconds after passing through a fixed point O is given by $v = 8t - t^2$. Find
(a) the initial acceleration, in ms^{-2}, of the particle,
(b) the acceleration, in ms^{-2}, when the particle stops instantaneously for the second time,
(c) the time, in seconds, when the velocity is uniform.

2. A particle moves along a straight line so that t seconds after passing through O, its velocity, v ms^{-1}, is given by $v = t^2 - 2t - 8$. Find
(a) the time, in seconds, when the acceleration of the particle is zero,
(b) the range of values of t, in seconds, when the particle experiences deceleration.

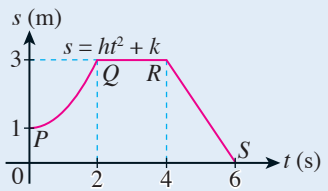
Formative Exercise 8.2

Quiz bit.ly/3IIudID

1. The diagram on the right shows a displacement function, $s = f(t)$, a velocity function, $v = f(t)$ and an acceleration function, $a = f(t)$ for a particle which moves along a straight line and passes through a fixed point O for $0 \leqslant t \leqslant 4$. Based on the graphs, determine
(a) the initial velocity of the particle, in ms^{-1},
(b) the time, in seconds, when the particle passes through the fixed point O,
(c) the minimum displacement, in m, of the particle,
(d) the total distance travelled, in m, by the particle in the given time period,
(e) the range of time, in seconds, when a particle moves to the right,



2. The diagram on the right shows a displacement-time graph of a particle moving along a straight line for t seconds. The equation of the curve PQ is $s = ht^2 + k$, where h and k are constants. The points P, Q, R and S are (0, 1), (2, 3), (4, 3) and (6, 0) respectively. Find
(a) the values of h and k,
(b) the instantaneous velocity, in ms^{-1}, of the particle when
(i) $t = 1$ (ii) $t = 3$ (iii) $t = 5$.



3. A particle moves along a straight line so that its displacement, s metre from a fixed point O at t seconds is given by $s = t^3 - 5t^2 - 8t + 12$, where $t \geqslant 0$.
(a) Express the velocity function, v and acceleration function, a of the particle in terms of t.
(b) Determine the instantaneous velocity, in ms^{-1}, and instantaneous acceleration, in ms^{-2}, of the particle when $t = 3$.
(c) Find the value of t, in seconds, when the particle is instantaneously at rest.
(d) Find the values of t, in seconds, when the particle is at O.
(e) Find the total distance travelled by the particle in the first 6 seconds.

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8.3 Integration in Kinematics of Linear Motion

Determining and interpreting the instantaneous velocity of a particle from its acceleration function

You have learnt that the acceleration function, a of a particle that moves linearly can be obtained by differentiating the velocity function, v with respect to time, t, that is:

$$a = \frac{dv}{dt}$$

Recall

Indefinite integral for a function $y = t^n$ with respect to t is $\int t^n \, dt = \frac{t^{n+1}}{n+1} + c$, where $n \neq -1$.

However, if the acceleration function, a of a particle is given, how can we determine the velocity function, v of the particle?

When the acceleration function $a = \frac{dv}{dt}$, the velocity function, v can be determined by integrating the acceleration function, a with respect to time, t which is $v = \int a \, dt$.

In general, the relationship between acceleration function, $a = h(t)$ and velocity function, $v = g(t)$ can be simplified as follows:

$$a = h(t) \rightarrow v = \int a \, dt \rightarrow v = g(t)$$

Example 11

A particle moves along a straight line and passes through a fixed point O with an initial velocity of 4 ms^{-1}. Its acceleration, a ms^{-2}, t seconds after passing through O is given by $a = 4 - 2t$.

(a) Calculate
  (i) the instantaneous velocity, in ms^{-1}, of the particle when $t = 7$,
  (ii) the maximum velocity, in ms^{-1}, of the particle,

(b) Find the possible values of t, in seconds, when the velocity of the particle is 7 ms^{-1}.

Solution

(a) (i) Given acceleration function, $a = 4 - 2t$.

So, velocity function, $v = \int (4 - 2t) \, dt$

$$v = 4t - t^2 + c$$

When $t = 0$ and $v = 4$,

So, $4 = 4(0) - (0)^2 + c$

$c = 4$

Thus, at time t, $v = 4t - t^2 + 4$.

When $t = 7$, $v = 4(7) - (7)^2 + 4$

$v = 28 - 49 + 4$

$v = -17$

Hence, the instantaneous velocity of the particle when $t = 7$ is -17 ms^{-1}.

CHAPTER 8

(ii) Maximum velocity, $\frac{dv}{dt} = 0$

So, $4 - 2t = 0$

$2t = 4$

$t = 2$

Since $\frac{d^2v}{dt^2} = -2 \ (< 0)$, v is maximum when $t = 2$.

Hence, maximum velocity of the particle

$= 4(2) - (2)^2 + 4$

$= 8 \text{ ms}^{-1}$

(b) When the instantaneous velocity of the particle is 7 ms^{-1},

$v = 7$

$4t - t^2 + 4 = 7$

$t^2 - 4t + 3 = 0$

$(t - 1)(t - 3) = 0$

$t = 1$ or $t = 3$

Thus, the possible values of t are 1 second and 3 seconds.

Information Corner

Minimum or maximum velocity occurs when $\frac{dv}{dt} = a = 0$ and depends on the values of $\frac{d^2v}{dt^2}$.

- If $\frac{d^2v}{dt^2} > 0$, then the velocity is minimum.
- If $\frac{d^2v}{dt^2} < 0$, then the velocity is maximum.

Self-Exercise 8.6

1. A particle moves along a straight line and passes through a fixed point O with an initial velocity of 10 ms^{-1}. The acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 4t - 8$, find

(a) the instantaneous velocity, in ms^{-1}, of the particle at the 4th second,

(b) the minimum velocity, in ms^{-1}, of the particle.

2. A particle moves from a fixed point O on a straight line with an initial velocity of 2 ms^{-1}. Its acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 4 - 6t$, find

(a) the instantaneous velocity, in ms^{-1}, of the particle when $t = 3$,

(b) the instantaneous velocity, in ms^{-1}, of the particle when $a = -8$.

3. A particle moves along a straight line at t seconds after passing a fixed point O. Its acceleration, a ms^{-2}, is given by $a = 6t - 24$. The particle passes through O with a velocity of 36 ms^{-1}. Find

(a) the range of values of t when the velocity is negative,

(b) the minimum velocity, in ms^{-1}, of the particle.

4. Stitching patterns on the straight edge of a head gear is done by a sewing machine. The initial velocity of the sewing machine is 20 cms^{-1}. The acceleration, in cms^{-2}, is given by $a = 8 - 2t$, where t is the time, in seconds, after passing through a fold. Calculate

(a) the instantaneous velocity of the sewing, in cms^{-1}, at the 2nd second,

(b) the instantaneous velocity of the sewing, in cms^{-1}, when the acceleration is zero,

(c) the time, in seconds, of the sewing when the acceleration is 5 cms^{-2},

(d) the value of t, in seconds, when the velocity of the sewing is 11 cms^{-1}.

KEMENTERIAN PENDIDIKAN MALAYSIA

Determining and interpreting the instantaneous displacement of a particle from its velocity and acceleration functions

Given a velocity function, v, how can we determine the displacement, s of the particle? How can we determine the velocity function, v and also the displacement function, s of a particle from an acceleration function, a?

When the velocity function, v is given as a function of time, t, the displacement function, s can be obtained by performing an integration, which is

$$s = \int v \, dt$$

and when the acceleration function, a is given as a function of time t, the displacement function, s can be obtained by performing two consecutive integrations, which are

$$v = \int a \, dt \quad \text{and} \quad s = \int v \, dt$$

Example 12

A particle moves along a straight line and passes through a fixed point O with a velocity of 12 ms^{-1}. The acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 4 - 2t$.

(a) Determine the instantaneous displacement, in ms^{-1}, of the particle from O

(i) when $t = 3$,

(ii) when the particle is at rest.

(b) Next, find the distance, in m, travelled by the particle in the 7th second.

Solution

Velocity function, v is given by $v = \int a \, dt$

$$v = \int (4 - 2t) \, dt$$

$$v = 4t - t^2 + c$$

When $t = 0$ and $v = 12$, then $12 = 4(0) - (0)^2 + c$.

$$c = 12$$

Hence, at time t, $v = 12 + 4t - t^2$.

Displacement function, s is given by, $s = \int v \, dt$

$$s = \int (12 + 4t - t^2) \, dt$$

$$s = 12t + 2t^2 - \frac{1}{3}t^3 + c$$

When $t = 0$ and $s = 0$, then, $0 = 12(0) + 2(0)^2 - \frac{1}{3}(0)^3 + c$.

$$c = 0$$

Hence, at time t, $s = 12t + 2t^2 - \frac{1}{3}t^3$

(a) (i) When $t = 3$, $s = 12(3) + 2(3)^2 - \frac{1}{3}(3)^3$

$$s = 36 + 18 - 9$$

$$s = 45$$

Hence, the instantaneous displacement when $t = 3$ is 45 m.

Excellent Tip

You are encouraged to draw a number line to illustrate the movement of a particle. When drawing the number line for the movement of a particle, for instance in the duration of $0 \leqslant t \leqslant n$, the following need to be labelled on the number line:

- the displacement of the particle when $t = 0$
- the time and the displacement, if available, of the particle when $v = 0$
- the displacement of the particle when $t = n$

Based on Example 12, draw a number line for the movement of a particle for $0 \leqslant t \leqslant 9$.

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(ii) When the particle is at rest, $v = 0$.

Then, $12 + 4t - t^2 = 0$

$(t + 2)(t - 6) = 0$

Since $t \geqslant 0$, $t = 6$,

When $t = 6$, $s = 12(6) + 2(6)^2 - \frac{1}{3}(6)^3$

$s = 72 + 72 - 72$

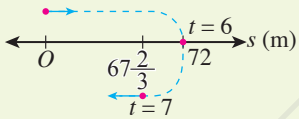
$s = 72$

Hence, the instantaneous displacement when the particle is at rest is 72 m.

(b) When $t = 7$, $s = 12(7) + 2(7)^2 - \frac{1}{3}(7)^3$

$= 84 + 98 - 114\frac{1}{3}$

$= 67\frac{2}{3}$



From the number line, the distance travelled by the particle in the 7th second $= |s_7 - s_6|$

$= \left|67\frac{2}{3} - 72\right|$

$= \left|-4\frac{1}{3}\right|$

$= 4\frac{1}{3}$ m

Information Corner

Time is one of scalar quantities. Scalar quantity is a physics quantity that has only magnitude. Thus, the value of time must be positive.

Excellent Tip

Total distance travelled in the first n seconds is the distance travelled by the particle from $t = 0$ to $t = n$. Whereas, distance travelled in the nth second is the distance travelled by the particle from $t = (n - 1)$ to $t = n$, that is $|s_n - s_{n-1}|$.

Self-Exercise 8.7

1. A particle moves along a straight line and passes through a fixed point O with an initial velocity of 3 ms^{-1}. Its acceleration, a ms^{-2}, t seconds after passing through O is given by $a = 6 - 3t$. Find the instantaneous displacement, in m, of the particle when
(a) $t = 5$, (b) its velocity is uniform.

2. Acceleration, a ms^{-2}, for a particle moving along a straight line, t seconds after passing through a fixed point O is given by $a = 12t - 8$. Given the velocity of the particle, $t = 1$ second after passing through O is -10 ms^{-1}. Find the instantaneous displacement, in m, of the particle when
(a) its acceleration is 4 ms^{-2}, (b) the particle is at rest.

3. A particle moves along a straight line and passes through a fixed point O with an initial velocity of 8 ms^{-1}. The acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 10 - 6t$, find
(a) the maximum displacement of the particle,
(b) the distance travelled by the particle in the 5th second.

4. Farhan participates in a cycling event organised by a cycling society. Farhan is cycling along the straight road. At t hours after passing the starting point, its acceleration, a kmh^{-1}, is given by $a = 4t - 16$ and his starting velocity is 30 kmh^{-1}.
(a) Express the acceleration function, s and the velocity function, v, in terms of t.
(b) Prove that Farhan stops instantaneously when $t = 3$.
(c) Find the total distance travelled, in km, by Farhan in the first 3 hours.

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Formative Exercise 8.3

Quiz bit.ly/33RoiEy

1. A particle passes a fixed point O with an initial velocity of 30 ms^{-1} and moves along a straight line with an acceleration $a = (12 - 6t)$ ms^{-2} where t is the time, in seconds, after passing through the point O.
(a) Calculate the velocity, in ms^{-1}, when $t = 2$,
(b) Where is the particle when $t = 1$?

2. A particle moves along a straight line and passes through a fixed point O. At t seconds after passing O, its velocity, v ms^{-1} is given by $v = 24t - 6t^2$. Calculate
(a) the initial acceleration, in ms^{-2}, of the particle,
(b) the value of t, in seconds, when the acceleration is zero,
(c) the value of t, in seconds, when the particle is at O again.

3. A particle moves along a straight line and passes through a fixed point O with a velocity of -12 ms^{-1} and an acceleration of -10 ms^{-2}. After t seconds of passing the fixed point O, the acceleration of the particle is $a = m + nt$, where m and n are constants. The particle stops instantaneously when $t = 6$. Calculate
[Assume motion to the right is positive.]
(a) the values of m and n,
(b) the minimum velocity, in ms^{-1}, of the particle,
(c) the total distance, in m, travelled by that particle in the first 9 seconds.

4. A particle moves along a straight line from a fixed point O. Its velocity, v ms^{-1}, at t seconds after leaving O is given by $v = 2t^2 - 5t - 3$. Calculate
(a) the displacement, in m, when the particle stops instantaneously,
(b) the range of time, in seconds, when the particle decelerates,
(c) the total distance, in m, travelled by the particle in the first 6 seconds.

5. Haiqal plays a remote control car along a straight track. The acceleration, a ms^{-2}, of the remote control car is given by $a = 12 - 4t$ where t is in seconds after the remote control car passes a fixed point O. Calculate
(a) the maximum velocity, in ms^{-1}, of the remote control car,
(b) the values of t, in seconds, when the velocity of the remote control car is zero,
(c) the distance travelled, in m, of the remote control car in the 5th second.

6. The diagram on the right shows Azlan running across a straight bridge in 25 seconds with a velocity, v ms^{-1}. His velocity, v ms^{-1}, t seconds after passing through M is given by $v = \frac{3}{4}t - \frac{3}{100}t^2$. Calculate
[Assume motion to the right is positive.]



(a) the value of t, in seconds, when the acceleration of Azlan is zero,
(b) the maximum velocity, in ms^{-1}, of Azlan,
(c) the total distance, in m, travelled by Azlan.

CHAPTER 8

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8.4 Applications of Kinematics of Linear Motion

Solve the kinematic problem of linear motion involving differentiation and integration

We have learned that the relationship between displacement, s, velocity, v and acceleration, a for an object that moves linearly is as follow.

Using differentiation	$v = \frac{ds}{dt}, a = \frac{dv}{dt}$	Using integration	$v = \int a\,dt, s = \int v\,dt$

With this knowledge and application skills, many problems involving linear motion can be solved.

Example 13 MATHEMATICAL APPLICATIONS

Fariza starts running along a straight lane for 30 seconds from a starting point. Her velocity, v ms^{-1}, after t seconds is given by $v = 0.9t - 0.03t^2$ where $0 \leqslant t \leqslant 30$. Find
(a) the time, in seconds, when her acceleration is zero,
(b) the distance travelled by Fariza, in metre.

Solution

1 . Understanding the problem

Given the velocity of Fariza is $v = 0.9t - 0.03t^2$ and when $t = 0$, $s = 0$, find
- the time when her acceleration is zero.
- the distance travelled in 30 seconds.

2 . Planning the strategy

- Use $a = \frac{dv}{dt}$ to determine the acceleration function and find the value of t when the acceleration is zero, $a = 0$.
- Use $s = \int v\,dt$ to determine the displacement function and substitute $t = 30$ in the displacement function to find the distance travelled by Fariza.

3 . Implementing the strategy

(a) Given $v = 0.9t - 0.03t^2$.

Then, $a = \frac{dv}{dt}$

$a = 0.9 - 0.06t$

When the acceleration is zero, $a = 0$.

$0.9 - 0.06t = 0$

$0.06t = 0.9$

$t = 15$

Hence, when $t = 15$, Fariza's acceleration is zero.

(b) $s = \int v\,dt$

$s = \int (0.9t - 0.03t^2)\,dt$

$s = 0.45t^2 - 0.01t^3 + c$

When $t = 0$ and $s = 0$ then, $c = 0$.

So, at time t, $s = 0.45t^2 - 0.01t^3$

When $t = 30$, $s = 0.45(30)^2 - 0.01(30)^3$

$s = 135$

Hence, Fariza ran a distance of 135 m in 30 seconds.

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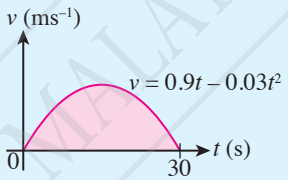
4 . Check and reflect

(a) Substitute $t = 15$ in the acceleration function, $a = 0.9 - 0.06t$ to make sure that Fariza's acceleration is zero when the time is 15 seconds.

$a = 0.9 - 0.06(15)$
$a = 0.9 - 0.9$
$a = 0$

(b) Sketch a velocity-time graph, $v = 0.9t - 0.03t^2$ for a time period $0 \leqslant t \leqslant 30$ and by using a definite integral, verify that the area under the graph for that time period is 135 m.

$$\text{Distance} = \int_0^{30} (0.9t - 0.03t^2)\, dt$$
$$= \left[0.45t^2 - 0.01t^3\right]_0^{30}$$
$$= [0.45(30)^2 - 0.01(30)^3] - [0.45(0)^2 - 0.01(0)^3]$$
$$= |135 - 0|$$
$$= 135 \text{ m}$$

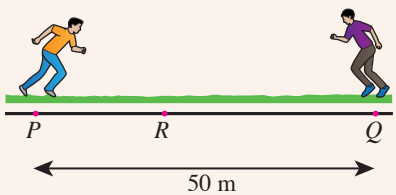


Self-Exercise 8.8

1. SMK Seri Aman launched a water rocket in a school field during the officiating ceremony of the Mathematics and Science Carnival. The rocket was launched vertically upward from the surface of the field with its velocity, v ms^{-1}, is given by $v = 20 - 10t$ after t seconds from the surface of the field. The rocket stops momentarily at p seconds.

(a) Find the value of p.
(b) Express in terms of t, the displacement, s metre, of the rocket at t seconds.
(c) Determine
  (i) the maximum height, in metre, of the rocket.
  (ii) the time, in seconds, when the rocket touches the surface of the field.

2. The diagram on the right shows the positions and directions of two boys, Faiz and Qian Hao running on a straight path, each passing two fixed points, P and Q at the same time. Faiz stops instantaneously at point R. The velocity of Faiz, v ms^{-1}, at t seconds after passing through the fixed point P is given by $v = 6 + 4t - 2t^2$ while Qian Hao runs with a constant velocity of -5 ms^{-1}. It is given that the distance PQ is 50 m.



[Assume motion to the right is positive.]

(a) Calculate the maximum velocity of Faiz, in ms^{-1}.
(b) (i) Sketch a velocity-time graph for Faiz from point P to point R.
  (ii) Then, find the distance travelled by Faiz, in m, from point P to point R.
(c) Determine the distance, in m, between Faiz and Qian Hao when Faiz is at point R.

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3. Azim runs along a straight path from a fixed point O. His velocity, v kmh^{-1}, t hours after passing through O is given by $v = mt^2 + nt$, where m and n are constants. Azim stops to rest after running half of the distance when $t = 1$ with an acceleration of 12.5 kmh^{-1}, find [Assume motion to the right is positive.]
(a) the value of m and of n, (b) the maximum velocity of Azim, in kmh^{-1},
(c) the distance, in km, travelled by Azim in the 2nd hour.

4. The diagram shows the movement of a car along a straight road starting from a fixed point O and heading towards point A and point B. The velocity, v ms^{-1}, of the car at t seconds after passing through the fixed point O is given by $v = 3t^2 - 16t - 12$. Given that the car is at point A when $t = 5$ and rests for a while at point B. Calculate
(a) the acceleration of the car, in ms^{-2}, at point B,
(b) the distance, in m, of AB.



Formative Exercise 8.4

Quiz bit.ly/2Fmh0zl

1. A cricket player hits a ball and it travels along a straight path through a centre P with a velocity of 44 ms^{-1}. The acceleration, in ms^{-2}, at t seconds after passing through the centre P is given by $a = 12 - 6t$. Calculate
(a) the maximum velocity, in ms^{-1}, of the ball,
(b) the distance, in m, of the ball from the centre P when $t = 2$.

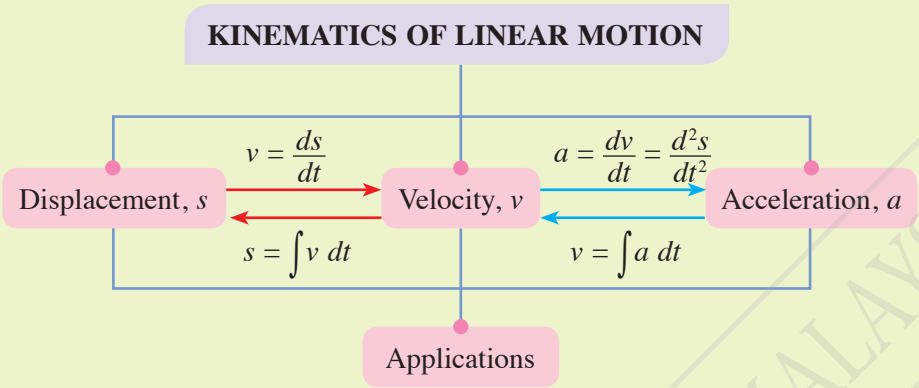
2. An object moves along a straight line from a fixed point X. Its acceleration, a ms^{-2}, at t seconds after passing the point X is given by $a = 16 - 4t$ for $0 \leqslant t \leqslant 3$. Given the velocity of the object at the time $t = 3$ is 38 ms^{-1}. Calculate
(a) the initial velocity, in ms^{-1}, of the object,
(b) the velocity, in ms^{-1}, of the object at the fourth second.

3. Objects A and B are placed on a horizontal straight line. A toy car moves along the straight line. The velocity, in ms^{-1}, of the toy car t seconds after passing through object A is given by $v = 2t - 4$. At the beginning of the movement, the toy car moves towards object B. [Assume motion to the right of the toy car is positive.]
(a) Calculate the range of values of t, in seconds, when the toy car moves towards the object B.
(b) Given that the distance between object A and object B is 5 m. Determine whether the toy car can reach object B or not.
(c) Find the total distance, in m, of the toy car in the first 6 seconds.
(d) Draw the displacement-time graph of the toy car from object A for $0 \leqslant t \leqslant 6$.

4. An experiment is conducted to study the motion of a particle along a straight line with a velocity, v ms^{-1}, t seconds from an initial point O. At t seconds after passing through O, the velocity, v ms^{-1}, is given by $v = 3t^2 - 8t + 4$. At the beginning of the experiment, the particle is 2 m to the right of O. Calculate
(a) the distance, in m, of the particle from O when $t = 5$,
(b) the minimum velocity, in ms^{-1}, of the particle,
(c) the range of time, in seconds, when the velocity of the particle is negative,
(d) the maximum displacement, in m, of the particle from the point O for $0 \leqslant t \leqslant 2$.

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REFLECTION CORNER



Notes	
• Initial displacement • Initial velocity • Initial acceleration	$t = 0$
• Minimum displacement • Maximum displacement	$v = 0$
• Minimum velocity • Maximum velocity	$a = 0$

Journal Writing

The techniques of differentiation and integration can be applied to determine displacement, velocity and acceleration of any object. Search the Internet and reference books for the application of differentiation and integration in the movement of objects. Then, create an interesting graphic folio.

CHAPTER 8

Summative Exercise

1. A particle moves along a straight line from a fixed point O. Its displacement, s metre, at t seconds after passing O is given by $s = 2t^3 - 24t^2 + 90t$. Calculate PL 3

(a) the distance, in m, of the particle from the fixed point O when $t = 8$,

(b) its velocity, in ms^{-1}, when $t = 1$,

(c) its acceleration, in ms^{-2}, when $t = 3$,

(d) the values of t, in seconds, when the particle stops momentarily.

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2. A particle moves along a straight line from a fixed point P for t seconds. Its displacement, s metre, at t seconds after passing P is given by $s = 3t^2 - 12t + 2$. Calculate PL 3

(a) the displacement, in metres, of the particle at $t = 3$,

(b) the initial velocity, in ms^{-1}, of the particle,

(c) its constant acceleration, in ms^{-2}.

3. Eleeza cycles passing her house to the shop along a straight pedestrian path. The displacement s metre, from her house at t minutes is given by $s = 2t^3 - 9t^2 + 12t + 6$ for $0 \leqslant t \leqslant 4$. PL 5

[Assume motion to the right is positive.]

(a) Calculate

(i) the initial velocity, in $mmin^{-1}$, of Eleeza,

(ii) the velocity, in $mmin^{-1}$, of Eleeza when $t = 3$,

(iii) the acceleration, in $mmin^{-2}$, of Eleeza when $t = 2$,

(iv) the distance, in m, travelled by Eleeza in the 7th minute.

(b) Sketch a velocity-time graph to represent Eleeza's journey for $0 \leqslant t \leqslant 4$.

4. A particle starts from O and moves along a straight line pass towards a point marked X whose displacement from O is 1.25 m. Its acceleration is given by 10 ms^{-2}. PL 4

(a) Determine the velocity function, v and the displacement function, a of the particle in terms of t.

(b) Find the time, in seconds, and the velocity, in ms^{-1}, when the particle is at the point X.

5. A particle moves along a straight line from a fixed point O for t seconds with an initial velocity of 8 ms^{-1}. The acceleration, a ms^{-2}, of the particle t seconds after leaving O is given by $a = 6 - 6t$. Calculate PL 3

[Assume motion to the right is positive.]

(a) the velocity, in ms^{-1}, of the particle when $t = 2$,

(b) the displacement, in m, of the particle from O when $t = 5$.

6. A particle moves along a straight line and passes through a fixed point O. The velocity, v ms^{-1}, of the particle t seconds after passing a fixed point O is given by $v = t^2 - 4t + 3$. Calculate PL 4

[Assume motion to the right is positive.]

(a) the values of t when the particle stops momentarily,

(b) the distance, in metres, travelled by the particle for $0 \leqslant t \leqslant 8$.

7. A particle moves along a straight line from a fixed point P. Its acceleration, a ms^{-2}, at t seconds after leaving P is given by $a = mt + n$, where m and n are constants. The particle moves with an initial velocity of 30 ms^{-1}, experiences a deceleration of 20 ms^{-2} and stops when $t = 2$. PL 5

[Assume motion to the right is positive.]

(a) Find the value of m and of n.

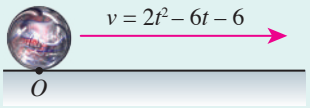
(b) Express the displacement function, s of the particle in terms of t.

(c) Find the value of t, in seconds, when the particle stops for the second time.

(d) Calculate the total distance, in m, of the particle travelled in the 2nd second.

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8. A marble moves along a straight line t seconds after passing through O. Its velocity, v ms^{-1}, is given by $v = 2t^2 - 6t - 6$. PL 3



(a) Calculate the velocity, in ms^{-1}, of the marble when $t = 2$.
(b) Find the acceleration, in ms^{-2}, of the marble when $v = 14$ ms^{-1}.

9. Irma drives along a straight road after leaving the car park at a shopping complex. The velocity, v ms^{-1}, of her car is given by $v = \frac{1}{2}t^2 - 2t$ where t is the time in seconds after passing the automatic bar. The initial displacement of the car is 50 metres. PL 2
(a) Calculate the value of t, in seconds, when the car stops instantaneously.
(b) Find the total distance, in m, of the car travelled in the first 7 seconds.
(c) Describe the movement of the car in the first 6 seconds.

10. A particle moves along a straight line and passes through a fixed point O. The velocity, v ms^{-1}, of the particle t seconds after passing through O is given by $v = t^2 - 8t$. PL 4
(a) Show that the maximum velocity, in ms^{-1}, of the particle is not zero.
(b) Find the displacement, to the nearest metre, travelled by the particle from the fixed point O when $t = 4$.

11. A particle moves along a straight line from a fixed point O. The displacement, s metre, of the particle t seconds after passing through O is given by $s = t^3 - 3t + 1$.
[Assume the movement to the right is positive.] PL 4
(a) Express the velocity and the acceleration in terms of t.
(b) Describe the motion of particles when $t = 0$ and $t = 2$.
(c) Find the time interval, in seconds, in which the particle changes in direction.

12. A particle moves along a straight line from an initial point. Its velocity, v ms^{-1}, at t seconds after passing through the initial point is given by $v = ht^2 + kt$ where h and k are constants. The particle stops instantaneously after 3 seconds with an acceleration at that time of 9 ms^{-2}. Find
[Assume the movement to the right is positive.] PL 5
(a) the values of h and k,
(b) the time, in seconds, when the particle returns to the initial point,
(c) the acceleration, in ms^{-2}, when the particle returns to the initial point,
(d) the total distance, in m, travelled by the particle in the first 5 seconds.

13. A particle moves along a straight line and passes through a fixed point O with a velocity of -6 ms^{-1}. Its acceleration, a ms^{-2}, at t seconds after passing through O is given by $a = 8 - 4t$.
[Assume the movement to the right is positive.] PL 5
(a) Find the maximum velocity, in ms^{-1}, of the particle.
(b) Find the time, in seconds, of the particle when it passed the fixed point O again.
(c) Sketch the velocity-time graph for the movement of the particle for $0 \leqslant t \leqslant 3$.
(d) Then, find the total distance, in m, travelled by the particle in the first 3 seconds.

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14. Teacher Azizah conducted an experiment to determine the velocity of the trolley along a straight track. The velocity, v cms^{-1}, of the trolley after passing the fixed point O is given by $v = t^2 - 7t + 6$. PL 5

[Assume the movement to the right is positive.]

(a) Find

(i) the initial velocity, in cms^{-1}, of the trolley,

(ii) the range of time, in seconds, when the trolley is moving to the left,

(iii) the range of time, in seconds, when the acceleration of the trolley is positive.

(b) Sketch the velocity-time for the movement of the trolley for $0 \leqslant t \leqslant 6$.

15. A particle moves along a straight line and passes through a fixed point O. The velocity, v ms^{-1}, t seconds after passing through O is given by $v = t^2 - 6t + 8$. The particle stops instantaneously at points P and R. PL 5

[Assume motion to the right is positive.]

(a) Find the minimum velocity, in ms^{-1}, of the particle.

(b) Calculate the distance, in m, between the point P and the point R.

(c) Sketch the velocity-time graph for $0 \leqslant t \leqslant 7$. Then, determine the range of values of t when the velocity of the particle is increasing.

MATHEMATICAL EXPLORATION

Instructions:

1. Divide the class into groups of four.

2. Each group is given a toy car. The toy car will be moved from the starting point marked **X**. Suppose the records of the movement of the toy car involves a straight path as below.



3. Each group needs to make a simulation for each instruction below.

(a) State the position of the toy car from the starting point **X** when the displacement is

(i) positive (ii) zero (iii) negative

(b) State whether the velocity of the toy car is positive or negative when the car moves from

(i) **X** to **B** (ii) **B** to **D** (iii) **D** to **A**

(iv) **A** to **C** (v) **C** to **X**

(c) State the velocity of the toy car when

(i) it stops at **C**, (ii) it changes direction at **D**.

(d) By moving the toy car, discuss in your group the meaning of acceleration, deceleration and zero acceleration.

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Answers

Open the Full Solutions file from the QR code on page (vii) to get the steps to the solution.

CHAPTER 1 CIRCULAR MEASURE

Self-Exercise 1.1

1. (a) 22.5° (b) 135° (c) 28° 39′ (d) 59° 35′

2. (a) $\frac{1}{10}\pi$ rad (b) $\frac{2}{3}\pi$ rad (c) $1\frac{1}{4}\pi$ rad (d) $1\frac{2}{3}\pi$ rad

Formative Exercise 1.1

1. (a) 105° (b) 240° (c) 114° 35′ (d) 274° 59′

2. (a) 1.327 rad (b) 2.426 rad (c) 3.535 rad (d) 5.589 rad

3. (a) 1.274 rad (b) 2.060 rad (c) 2.627 rad (d) 3.840 rad

Self-Exercise 1.2

1. (a) 13.2 cm (b) 16 cm (c) 13.09 cm (d) 6.92 cm

2. (a) 5 cm (b) 6.42 cm

3. (a) 2.002 rad (b) 10.01 cm

Self-Exercise 1.3

1. (a) 26.39 cm (b) 20.47 cm
(c) 30.62 cm (d) 32.74 cm

2. (a) 114° 35′ (b) 25.78 cm

Self-Exercise 1.4

1. (a) 34.96 cm (b) 7.25 cm (c) 39.87 cm

2. 5 663.819 km **3.** 37.1 m

4. (a) 109.97 cm (b) 379.97 cm

5. 89.66 cm

Formative Exercise 1.2

1. (a) 1.484 rad (b) 10.11 cm

2. 0.7692 rad

3. (a) 0.6435 rad (b) 7.218 cm

4. (a) 4 cm (b) 2 cm

5. (a) 8.902 cm (b) 18.44 cm

6. 26.39 cm

7. (a) 103.686 m (b) 2 073.72 m

Self-Exercise 1.5

1. (a) 19.8 cm^2 (b) 107.5 cm^2
(c) 13.09 cm^2 (d) 471.4 cm^2

2. 15 cm^2

3. (a) 10 cm (b) 39 cm (c) 59 cm

4. (a) 1.2 rad (b) 12 cm (c) 32 cm

Self-Exercise 1.6

1. (a) 12.31 cm^2 (b) 61.43 cm^2
(c) 2.049 cm^2 (d) 42.52 cm^2

2. (a) 95° 30′ (b) 3.023 cm^2

3. (a) 1.047 rad (b) 1.448 cm^2

Self-Exercise 1.7

1. (a) 75.70 m (b) 114.22 m^2

2. (a) 4.063 cm (b) 50.67 cm^2

3. (a) 77° 10′ (b) 32.48 cm^2

4. (a) 67.04 cm^2 (b) 2.5 rad

Formative Exercise 1.3

1. (a) 0.7 rad (b) 10.35 cm^2

2. (a) 1.047 rad (b) 2.263 cm^2

3. (a) 3.77 rad (b) 47.13 cm^2

4. (a) 1.75 rad (b) 36.27 cm^2

5. (a) 24.73 cm (b) 222.57 cm^2
(c) 98.98 cm^2 (d) 123.59 cm^2

6. (b) 34.44 cm^2 (c) $n = 5$, 16.46 cm^2

Self-Exercise 1.8

1. (a) 1.855 rad, 1.75 rad (b) 132.37 cm
(c) 349.18 cm^2

2. 8.931 mm

Formative Exercise 1.4

1. (a) (i) 29.68 cm (ii) 42.23 cm^2 (iii) 337.84 cm^3
(b) 1 350 grams

2. (a) 40.96 m (b) 109.156 m^2 (c) 163.734 m^3

3. (a) 1.344 rad (b) 61.824 cm (c) 391.068 cm^2

4. (a) (i) 31.41 cm (ii) 471.15 cm^2
(iii) 61.41 cm (iv) 81.44 cm^2
(b) 7 067.25 cm^3 (c) RM3 533.63

Summative Exercise

1. (a) 1.2 rad (b) 32 cm

2. (a) 23.049 cm (b) 31.908 cm^2

3. (a) 1.08 rad (b) 14.8 cm

4. (a) $2r + r\theta = 18$, $\frac{1}{2}r^2\theta = 8$ (b) $r = 8$ cm, $\theta = \frac{1}{4}$ rad

5. (a) 16° 16′ (b) 3.42 cm (c) 0.45 cm^2

6. (a) 0.6284 rad (b) 71.87 cm^2

7. $0.433r^2$ **8.** 60.67 cm

9. (a) 8 cm (b) 55.44 cm^2 (c) 5.791 cm^2

10. (a) 25 units^2 (b) 90° (c) 25 units^2

11. (a) 2.636 rad (b) 21.09 units^2 (c) 13.34 units^2

12. (a) 6.711 cm (b) 39.50 cm
(c) 24.5 cm^2 (d) 77.80 cm^2

13. (a) 6.282 cm (b) 3.54 cm^2

14. (a) 1.5 rad (b) 65.55 m (c) 155.07 m^2

15. 78.564 cm

16. (b) (i) 1 261.75 cm^2 (ii) 720.945 cm^2
(iii) 144.189 litres

17. (a) 2.094 cm (b) 3.141 cm^2
(c) 12.564 cm^3 (d) 38.658 cm^2

18. (a) 62.82 cm (b) 27.12 cm^2

CHAPTER 2 DIFFERENTIATION

Self-Exercise 2.1

1. (a) –3 (b) 1 (c) –2 (d) 1

2. (a) –1 (b) 4 (c) –5 (d) $\frac{1}{12}$ (e) $\frac{1}{4}$
(f) 1 (g) 4 (h) $-\frac{1}{3}$ (i) $\frac{4}{5}$

3. (a) $\frac{1}{2}$ (b) $\frac{2}{7}$ (c) 1
(d) –30 (e) 4 (f) $\frac{1}{6}$

4. (a) (i) 4 (ii) Does not exist
(b) (i) 2 (ii) 3

Self-Exercise 2.2

1. (a) 1 (b) 5 (c) –4 (d) $12x$
(e) $-2x$ (f) $6x^2$ (g) x (h) $-\frac{1}{x^2}$

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2. $4x - 1$ **3.** $1 - 2x$

Formative Exercise 2.1

1. (a) (i) 8 (ii) 3 (iii) 0
(iv) -1 (v) 0 (vi) 3
(b) $-1, 5$
(c) (i) $2x - 4$ (ii) 4

2. (a) 9 (b) 2 (c) $-\frac{1}{18}$
(d) 3 (e) 2 (f) $\frac{3}{10}$

3. (a) 2 (b) $-\frac{1}{6}$ (c) -4

4. (a) $k = 4$ (b) 5

5. (a) 5 (b) $2x - 1$ (c) $2x + 2$ (d) $-\frac{1}{4x^2}$

6. 7 ms^{-1}

Self-Exercise 2.3

1. (a) $8x^9$ (b) $-8x^3$ (c) $-\frac{6}{x^9}$
(d) $-\frac{2}{\sqrt[3]{x^4}}$ (e) $-\frac{8}{\sqrt[3]{x}}$

2. (a) $8x + 6$ (b) $\frac{2}{5\sqrt{x}} - \frac{1}{\sqrt{x^3}}$ (c) $32x - 72$

3. (a) $40x - 10\sqrt{x^3}$ (b) $4x^3 + 8 - \frac{32}{x^3}$
(c) $\frac{5}{2\sqrt{x}} - 6\sqrt{x} + \frac{1}{2\sqrt{x^3}}$

4. (a) -1 (b) $-4\frac{1}{6}$ (c) -1

Self-Exercise 2.4

1. (a) $5(x + 4)^4$ (b) $8(2x - 3)^3$
(c) $-6(6 - 3x)^5$ (d) $56x(4x^2 - 5)^6$
(e) $\frac{4}{3}\left(\frac{1}{6}x + 2\right)^7$ (f) $-12(5 - 2x)^8$
(g) $-3(2x + 1)(1 - x - x^2)^2$ (h) $-\frac{20(3x^2 - 2)}{(2x^3 - 4x + 1)^{11}}$

2. (a) $-\frac{3}{(3x + 2)^2}$ (b) $-\frac{6}{(2x - 7)^4}$
(c) $\frac{100}{(3 - 4x)^6}$ (d) $-\frac{30}{(5x - 6)^9}$
(e) $\frac{1}{\sqrt{2x - 7}}$ (f) $-\frac{3}{2\sqrt{6 - 3x}}$
(g) $\frac{3x}{\sqrt{3x^2 + 5}}$ (h) $\frac{2x - 1}{2\sqrt{x^2 - x + 1}}$

3. (a) 2 744 (b) $-\frac{1}{2}$ (c) -2

Self-Exercise 2.5

1. (a) $60x^2 + 24x$ (b) $-8x^3 - 6x^2$
(c) $2x(1 - 12x)(1 - 4x)^3$ (d) $\frac{2x(1 - 3x^2)}{\sqrt{1 - 2x^2}}$
(e) $8(7x - 1)(2x + 7)^5$ (f) $(7x + 8)(x + 5)^2(x - 4)^3$

2. (a) $-2(9x^2 + x - 3)$ (b) $3x^2 + 2 + \frac{4}{x^3}$
(c) $5x^4 - 8x^3 + 24x^2 - 10x + 10$

3. $\frac{13}{4}$ **4.** $\frac{41}{5}$

5. (a) $-\frac{6}{(2x - 7)^2}$ (b) $\frac{18}{(4x + 6)^2}$
(c) $\frac{8x(1 - 3x)}{(1 - 6x)^2}$ (d) $\frac{4x^3 - 3x^2 - 2}{(2x - 1)^2}$
(e) $\frac{1 - x}{2\sqrt{x}\,(x + 1)^2}$ (f) $\frac{x - 2}{2\sqrt{(x - 1)^3}}$
(g) $\frac{6x(x^2 + 3)}{\sqrt{(2x^2 + 3)^3}}$ (h) $-\frac{6x^2 + 3x + 14}{\left(\sqrt{4x + 1}\right)\sqrt{(3x^2 - 7)^3}}$

6. 13

Formative Exercise 2.2

1. (a) $18x + \frac{6}{x^3}$ (b) $\frac{1}{x^2} - \frac{18}{x^4}$
(c) $5 + \frac{2}{\sqrt{x}}$ (d) $-\frac{5}{\sqrt{x^3}} - \frac{1}{\sqrt[3]{x^4}}$
(e) $4x^3 - 6 - \frac{18}{x^3}$ (f) $12\sqrt{x} + \frac{1}{2\sqrt{x}}$
(g) $-\frac{4}{3x^4} - \pi$ (h) $\frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}$

2. $\frac{7}{8}$

3. (a) $6t^{\frac{8}{3}}$ (b) $16t^{\frac{5}{3}}$ (c) $\frac{1}{2}$

4. $6t + 5, t < -\frac{5}{6}$

5. $a = 5, b = -4$

6. (1, 6)

7. (a) $h'(x) = 3kx^2 - 8x - 5$ (b) 7

8. (a) $\frac{1}{2}\left(\frac{x}{6} - 1\right)^3$ (b) $5(10x - 3)^5$
(c) $\frac{40}{(2 - 5x)^2}$ (d) $3\left(1 + \frac{1}{x^2}\right)\left(x - \frac{1}{x}\right)^2$
(e) $\frac{3}{\sqrt[3]{(3 - 9x)^4}}$ (f) $\frac{x + 3}{\sqrt{x^2 + 6x + 6}}$

9. -144 **10.** $a = 9, b = 4$

11. (a) $4(12x - 1)(2x - 1)^4$ (b) $x^3(33x + 4)(3x + 1)^6$
(c) $\frac{3(x + 2)}{2\sqrt{x + 3}}$ (d) $4(2x - 1)(x + 7)^4(x - 5)^2$
(e) $-\frac{1}{\sqrt{x}\left(1 + \sqrt{x}\right)^2}$ (f) $\frac{2x + 1}{\sqrt{(4x + 1)^3}}$
(g) $-\frac{2(x + 1)}{(x^2 + 2x + 7)^2}$ (h) $\frac{6x^2 - 4x^3 - 1}{(x - 1)^2}$

13. $\frac{4 + 6x - 4x^2}{(x^2 + 1)^2}, \frac{3}{4} < x < 2$

14. $x < -1$

Self-Exercise 2.6

1. (a) $12x^3 - 10x + 2, 36x^2 - 10$
(b) $8x + \frac{2}{x^2}, 8 - \frac{4}{x^3}$
(c) $24(3x + 2)^7, 504(3x + 2)^6$

2. (a) $\frac{1}{2\sqrt{x}} - \frac{2}{x^3}, -\frac{1}{4x^{\frac{3}{2}}} + \frac{6}{x^4}$ (b) $2x - \frac{4}{x^3}, 2 + \frac{12}{x^4}$
(c) $-\frac{7}{(x - 1)^2}, \frac{14}{(x - 1)^3}$

3. $(-3, 29)$ and $(1, -3), -12, 12$

Formative Exercise 2.3

2. (a) $-3, -12$ (b) 9, 24 (c) 0, 2

3. $\frac{3}{2}, -\frac{5}{8}$ **4.** $-\frac{1}{3}, 1$ **5.** 2

6. (a) $-\frac{4}{3}, 2$ (b) $6x - 2$ (c) $\frac{1}{3}$ (d) $x < \frac{1}{3}$

Self-Exercise 2.7

1. (a) (i) $-7, 8$
(ii) At $x = \frac{1}{4}$, the tangent line slants to the left.
At $x = 1$, the tangent line slants to the right.
(b) $\left(\frac{1}{3}, 6\right), \left(-\frac{1}{3}, -6\right)$

2. (a) $a = 2, b = 4$
(b) (1, 6)

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Self-Exercise 2.8

1. (a) $y = 3x - 6, 3y + x + 8 = 0$
(b) $y = 7x - 10, 7y + x = 30$
(c) $3y - x = 5, y = -3x + 15$
(d) $2y = -x + 7, y = 2x - 4$

2. (a) $y = 2x - 1, 2y + x = 3$
(b) $16y - 5x = 4, 10y = -32x + 143$
(c) $y = \frac{1}{4}x + \frac{5}{4}, y = -4x + 14$
(d) $5y - 4x = 13, 4y + 5x + 6 = 0$
(e) $y = -x, y = x + 2$
(f) $y = \frac{3}{4}x + \frac{3}{4}, y = -\frac{4}{3}x + 7$

3. (a) $\frac{13}{3}$ (b) $3y - 13x = 16$
(c) $13y + 3x + 168 = 0$

4. (a) 6 (b) $A(14, 0)$

Self-Exercise 2.9

1. (a) $y + x = 3$ (b) $3y + x = 15$ (c) $C(-3, 6)$

2. (a) $y = x - 6$ (b) $B(2, -4)$ (c) $M_{AB} = \left(\frac{3}{2}, -\frac{9}{2}\right)$

3. (a) $a = \frac{1}{2}, b = 5$ (b) $2y + x = 4$
(c) $R(4, 0)$ (d) $1\frac{1}{4}$ units^2

4. (a) $a = 1, b = 4$ (b) $y + 3x = 8$
(c) $Q\left(6, 6\frac{2}{3}\right)$ (d) $M_{PQ} = \left(3\frac{1}{2}, 5\frac{5}{6}\right)$

5. (a) $3\sqrt{10}$ units (b) $h = \frac{1}{2}, k = -2$

Self-Exercise 2.10

1. (a) $(-2, 16)$ is a maximum point,
$(2, -16)$ is a minimum point.
(b) $(2, 32)$ is a maximum point,
$(6, 0)$ is a minimum point.
(c) $(3, 9)$ is a maximum point,
$(-3, -9)$ is a minimum point.
(d) $(4, 8)$ is a maximum point.
(e) $(-2, -4)$ is a maximum point,
$(2, 4)$ is a minimum point.
(f) $(1, 2)$ is a minimum point.
(g) $(0, -1)$ is a maximum point,
$(2, 3)$ is a minimum point.
(h) $(-3, -12)$ is a maximum point,
$(3, 0)$ is a minimum point.

2. (a) $2(2x - 1)(x - 2)^2$ (b) $P\left(\frac{1}{2}, -\frac{27}{16}\right)$ and $Q(2, 0)$
(c) Q is a point of inflection.

Self-Exercise 2.11

1. (b) 400 cm^2

2. (a) $y = 120 - 25x$
(c) (i) $x = 2\frac{2}{3}$ cm, $y = 53\frac{1}{3}$ cm
(ii) 3 840 cm^2

3. (b) The radius is 2 cm and the height is 8 cm

Self-Exercise 2.12

1. (a) 6 units s^{-1} (b) 6 units s^{-1} (c) -36 units s^{-1}
(d) 40 units s^{-1} (e) 2 units s^{-1} (f) 24 units s^{-1}

2. (a) -6 units s^{-1} (b) 2 units s^{-1} (c) 4 units s^{-1}
(d) -6 units s^{-1} (e) 18 units s^{-1} (f) 18 units s^{-1}

3. (a) $\frac{3x}{2\sqrt{x + 4}}$ (b) 15 units s^{-1}

Self-Exercise 2.13

1. 3 units s^{-1} **2.** 2 cms^{-1} **3.** $-\frac{7}{200}$ cmmin^{-1}

4. (a) $V = 9\pi h$ (b) -5.4π cm^{3}min^{-1}

5. (a) 1.5 ms^{-1} (b) 5 ms^{-1}

Self-Exercise 2.14

1. (a) 0.3 unit (b) -0.5 unit

2. (a) -0.05 unit (b) $2p$ unit

3. -4, 3.92 **4.** 3.2%

Self-Exercise 2.15

1. $\frac{\pi\sqrt{10}}{600}$ second **2.** 0.0025 cm

3. -0.12 cm^3 **4.** -2π cm^3

Formative Exercise 2.4

1. (a) $2y - x = 2, Q(-2, 0)$ (b) $y = -2x + 1, R\left(\frac{1}{2}, 0\right)$
(c) $1\frac{1}{4}$ units^2

2. (a) $a = 3, b = -2$ (b) $y = 2x - 8, B(4, 0)$
(c) $2y + x + 1 = 0, C(-1, 0)$ (d) 5 units^2

3. (b) 5 cm, 62.5 cm^3

4. (a) -4 ms^{-1} (b) 1.5 ms^{-1}

5. -8 ms^{-1}

Summative Exercise

1. (a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $k = \pm 3$

2. -4

3. (a) $-\frac{2}{(2x + 1)^2}$ (b) $4(12x - 1)(2x - 1)^4$
(c) $\frac{12}{(2 - x)^3}$ (d) $\frac{3(x + 2)}{2\sqrt{x + 3}}$

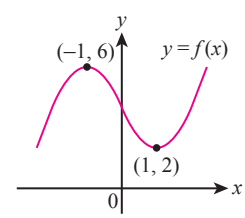
4. (a) $12 - 3x$ (b) 4

5. $a = 3, b = -\frac{1}{2}$ **6.** 5 cm

7. (a) -0.0735 unit (b) 1.927

8. -1% **9.** $1.6p\%$

10. (a) The maximum point is $(-1, 6)$ and the minimum point is $(1, 2)$
(b)



11. (a) $y = 32x - 63$ (b) $(-2, -14)$

12. (a) 6 cm (b) 144π cm^3

13. 40 m **14.** 48 cm^{2}s^{-1}

15. (b) (i) 12 units^2 s^{-1} (ii) 15 units^2 s^{-1}

16. (b) (i) -0.09π cm^3 (ii) Decrease $3p\%$

CHAPTER 3 INTEGRATION

Self-Exercise 3.1

1. $5x^3 + 4x$ **2.** $8x^3$

3. (a) $300t^2 + 60t$
(b) 4 600 litres

Formative Exercise 3.1

1. $18(2x + 2)^2, 3(2x + 2)^3$ **2.** $\frac{16}{(2 - 3x)^2}, \frac{5x + 2}{2 - 3x}$

3. 17, 32 **4.** $\frac{1}{3}$

5. (a) RM4 750
(b) Company K

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Self-Exercise 3.2

1. (a) $2x + c$ (b) $\frac{5}{6}x + c$ (c) $-2x + c$ (d) $\frac{\pi}{3}x + c$

2. (a) $x^3 + c$ (b) $\frac{x^4}{3} + c$ (c) $-\frac{x^2}{2} + c$
(d) $\frac{2}{x} + c$ (e) $-\frac{3}{2x^2} + c$ (f) $2\sqrt{x^3} + c$
(g) $3\sqrt[3]{x^2} + c$ (h) $\frac{54}{\sqrt{x}} + c$

3. (a) $x^2 + 3x + c$ (b) $\frac{4}{3}x^3 + \frac{5}{2}x^2 + c$
(c) $\frac{1}{8}x^4 + \frac{5}{2}x^2 - 2x + c$ (d) $-\frac{3}{x} + 2x^2 - 2x + c$

4. (a) $\frac{x^3}{3} - x^2 - 8x + c$ (b) $\frac{3}{5}x^5 + \frac{5}{4}x^4 + c$
(c) $\frac{5}{3}x^3 - 2\sqrt{x^3} + c$ (d) $\frac{25}{3}x^3 - 15x^2 + 9x + c$
(e) $\frac{5}{2}x^2 - 3x + c$ (f) $\frac{1}{3}x^3 + \frac{4}{5}x^{\frac{5}{2}} + \frac{1}{2}x^2 + c$

Self-Exercise 3.3

1. (a) $\frac{(x-3)^2}{3} + c$ (b) $\frac{(3x-5)^{10}}{30} + c$
(c) $\frac{2}{15}(5x-2)^6 + c$ (d) $\frac{(7x-3)^5}{105} + c$
(e) $-\frac{3}{(2x-6)^2} + c$ (f) $-\frac{2}{9(3x-2)} + c$

2. (a) $\frac{(4x+5)^5}{20} + c$ (b) $\frac{(3x-2)^4}{6} + c$
(c) $\frac{(5x-11)^5}{25} + c$ (d) $\frac{(3x-5)^6}{90} + c$
(e) $-\frac{1}{6(6x-3)^5} + c$ (f) $-\frac{4}{7(3x-5)^7} + c$

Self-Exercise 3.4

1. (a) 3 (b) 6

2. $\frac{33}{16}$

3. (a) $y = 3x^3 - 2x + 5$ (b) $y = 5x^2 - 2x - 3$
(c) $y = 8x^3 - 5x - 2$ (d) $y = 6x^3 + 5x^2 + 18$

Formative Exercise 3.2

1. (a) $\frac{1}{2}x + c$ (b) $-\frac{5}{6x^2} + c$
(c) $2x^{\frac{1}{2}} + c$ (d) $-\frac{1}{x^2} + \frac{1}{x^3} + c$

2. (a) $\frac{5}{2}x^2 - x^3 + c$ (b) $\frac{3}{2}x^2 + x + c$
(c) $-\frac{(5-6x)^4}{24} + c$ (d) $-\frac{2(5-2x)^{\frac{3}{4}}}{3} + c$

3. $p = 2, y = 21$

4. (a) 60 (b) $x = 0, -2$

5. $y = x^3 - 4x^2 + 2$ **6.** $y = 2x - 3x^2 + 10$

7. $a = 6, b = 5, y = 3x^2 + 5x + 6$

8. 44 m

Self-Exercise 3.5

1. (a) 60 (b) $\frac{3}{2}$ (c) $\frac{356}{3}$
(d) $-\frac{287}{9}$ (e) 9.203 (f) 6.992

2. (a) $\frac{74}{3}$ (b) $\frac{16}{3}$ (c) $-\frac{108}{125}$
(d) 43 (e) $\frac{33}{6\,272}$ (f) 1.827

3. (a) –3 (b) $\frac{3}{2}$ (c) 3

4. (a) 12 (b) 5 (c) 45

Self-Exercise 3.6

1. (a) $\frac{21}{2}$ units^2 (b) $\frac{35}{6}$ units^2 (c) $\frac{33}{2}$ units^2

2. (a) $\frac{212}{3}$ units^2 (b) $\frac{4}{3}$ units^2 (c) $\frac{100}{3}$ units^2

3. (a) $\frac{5}{3}$ units^2 (b) 9 units^2

Self-Exercise 3.7

1. (a) $\frac{32}{5}\pi$ units^3 (b) 9π units^3

2. $\frac{2}{5}\pi$ units^3 **3.** $\frac{123}{5}\pi$ units^3

4. (a) $A(0, -2)$ (b) $B(3, 1)$ (c) $\frac{108}{5}\pi$ units^3

Formative Exercise 3.3

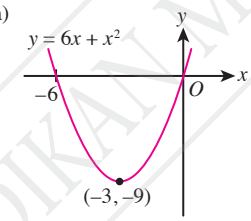
1. (a) $\frac{364}{3}$ (b) 5 (c) $\frac{155}{2}$

2. (a) 20 (b) 4

3. $h = 3$

4. (a) $K(1, 1)$ (b) 25 : 7

5. (a)



(b) $y = 6x, y = 10x - 4$ (c) $A(1, 6), \frac{2}{3}$ unit^2

6. $\frac{15}{2}\pi$ units^3

7. (a) $Q(0, 3)$ (b) $\frac{1}{3}$ unit^2 (c) 8π units^3

8. (a) $A\left(-\frac{1}{4}, \frac{5}{2}\right)$ (b) 0.027 unit^2 (c) $\frac{49}{32}\pi$ units^3

Self-Exercise 3.8

1. (b) 62 500π cm^3

2. (a) RM42 456 (b) 8.75%

Formative Exercise 3.4

1. 450 cm^3 **2.** RM119.98

3. (a) 350 (b) 66

Summative Exercise

1. (a) $\frac{1}{4}x^4 + \frac{1}{3}x^3 - 3x^2 + c$ (b) $-\frac{1}{2(2x-3)^2} + c$

2. (a) $a = -\frac{1}{3}, n = 3$ (b) $\frac{64}{49}$

3. $\frac{459}{76}$ **4.** $-\frac{21}{2}$

5. (a) 4 (b) $v = 5$

6. 138 cm^3

7. (a) $K(4, 1)$ (b) $\frac{8}{3}$ units^2

8. (a) $P(1, 9)$ (b) $\frac{10}{3}$ units^2 (c) 3 units^2

9. (a) $P(-3, 4)$ (b) $\frac{17}{3}$ units^2 (c) 30π units^3

10. (a) $P(0, 5), R\left(\frac{5}{2}, 0\right), S(0, 4)$
(b) $\frac{1}{3}$ unit^2 (c) $\frac{1}{2}\pi$ units^3

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11. $p = 3, q = 18$

12. (a) $\frac{257}{3}$ units^2 (b) 98π units^3

13. (a) $c = -2, A(2, 0)$ (b) $\frac{271}{6}$ units^2

(c) $\frac{92}{15}\pi$ units^3

14. 50.13 kg

15. (a) 300 m^3 (b) No

CHAPTER 4 PERMUTATION AND COMBINATION

Self-Exercise 4.1

1. 15 **2.** 30

3. (a) 20 (b) 240

Self-Exercise 4.2

1. (a) 336 (b) 55 (c) 6 (d) 4 200

2. (a) 24 (b) 120 (c) 720 (d) 362 880

3. 720 **4.** 2 520

Self-Exercise 4.3

1. (a) 60 (b) 40 320 (c) 15 120 (d) 5 040

2. 504 **3.** 60 **4.** 1 680 **5.** 25 200

Self-Exercise 4.4

1. (a) 360 (b) 840 (c) 90 720 (d) 60 540 480

2. 56 **3.** 210 **4.** 630

Self-Exercise 4.5

1. (a) 12 (b) 12 (c) 24

2. 300 **3.** 22 680 **4.** 42

Formative Exercise 4.1

1. 200

2. (a) 1 000 (b) 720

3. 24, 18

4. (a) 725 760 (b) 80 640 (c) 2 903 040

5. BAKU = 24, BAKA = 12
Not same because the word BAKA contains identical objects, which is A.

6. 56 **7.** 840

Self-Exercise 4.6

Combination because there is no condition on the sequence to choose the channel.

Self-Exercise 4.7

1. (a) 95 040 (b) 792

2. 2 300 **3.** 15 **4.** 20

Self-Exercise 4.8

1. 30 **2.** 45

3. (a) 15 (b) 65

Formative Exercise 4.2

2. (a) 56 (b) 30 (c) 16

3. 15 **4.** 45

5. (a) 34 650 (b) 924

Summative Exercise

1. 1 680, 1 050 **2.** 1 402 410 240

3. (a) 96 (b) 108

4. 243 **5.** 180 **6.** 360 360 **7.** 504

8. (a) 48 (b) 72

9. 1 155 **10.** 266

11. (a) 56 (b) 4 (c) 32

12. (a) 4 (b) 1 (c) 3

13. (a) 105 (b) 102

14. (a) 36 (b) 84 (c) 126

CHAPTER 5 PROBABILITY DISTRIBUTION

Self-Exercise 5.1

1. (a) {win, draw, lose} (b) {0, 1, 2, 3, 4, 5}
(c) {0, 1, 2, 3}

2. $X = \{0, 1, 2, 3, 4\}$

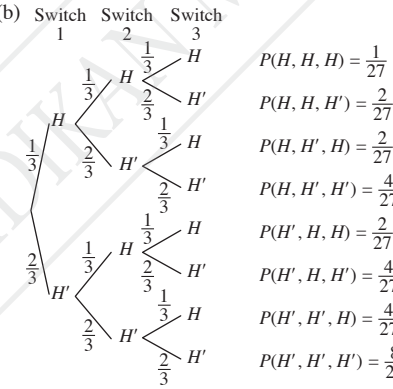
Self-Exercise 5.2

1. (a) $X = \{0, 1, 2, 3, 4, 5, 6\}$
Discrete random variable
(b) $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$
Discrete random variable
(c) $X = \{x : 3 \leqslant x \leqslant 460\}$
Continuous random variable

Self-Exercise 5.3

1. (a) $X = \{0, 1, 2, 3\}$

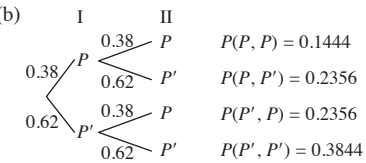
(b)



(c) 1

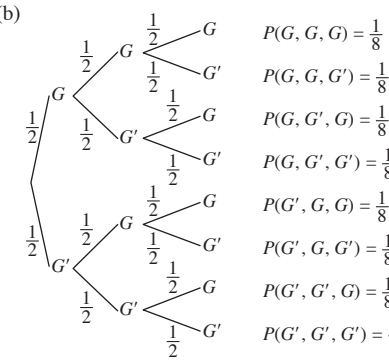
2. (a) $X = \{0, 1, 2\}$

(b)



3. (a) $X = \{0, 1, 2, 3\}$

(b)

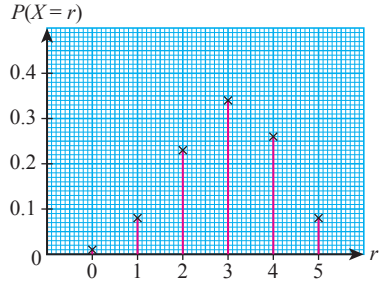


(c) $\sum_{i=1}^{8} P(X = r_i) = 1$

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Self-Exercise 5.4

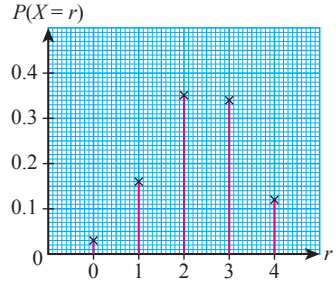
1. $P(X = r)$



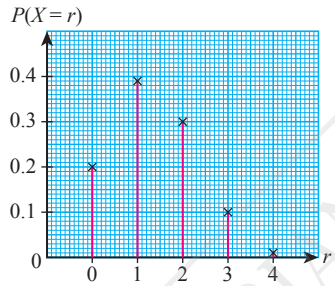
2. (a)

$X = r$	0	1	2	3	4
$P(X = r)$	0.0282	0.1627	0.3511	0.3368	0.1212

(b) $P(X = r)$



3. $P(X = r)$

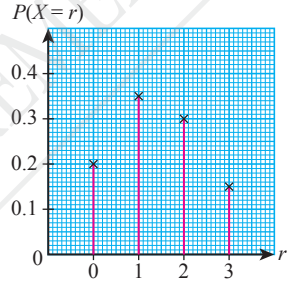


Formative Exercise 5.1

1. (a) $X = \{0, 1, 2\}$
(b) Discrete random variable

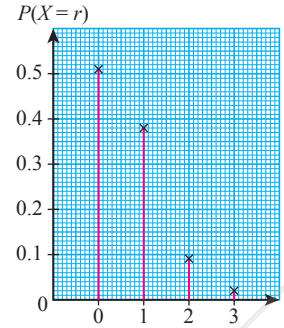
2. (a) $X = \{x : 1.2 \text{ cm} \leqslant x \leqslant 10.2 \text{ cm}\}$
(b) Continuous random variable

3. (b) $P(X = r)$



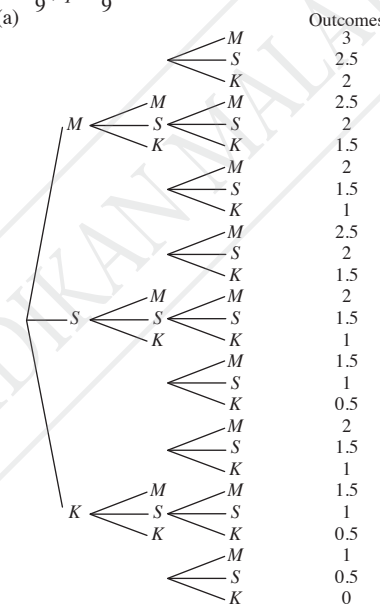
4. (a) $X = \{0, 1, 2, 3\}$

(c) $P(X = r)$



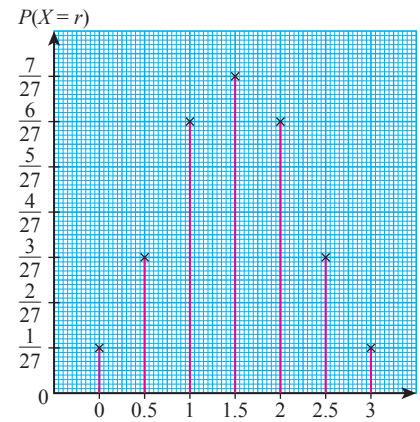
5. $p = \frac{2}{9}, q = \frac{1}{9}$

6. (a)



(b) $X = \{0, 0.5, 1, 1.5, 2, 2.5, 3\}$

(c) $P(X = r)$



Self-Exercise 5.5

1. (a) $X = (0, 1\}$ (b) 0.7

2. Not a binomial distribution.

3. Binomial distribution.

4. Yes

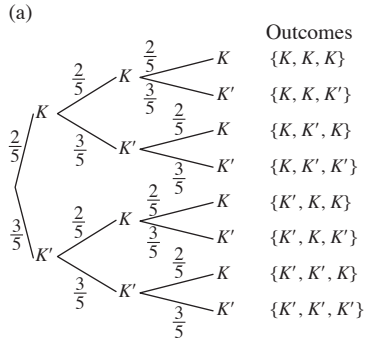
5. Not a binomial distribution.

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Self-Exercise 5.6

1. (a) 0.1776 (b) 0.0711

2. (a)



(b) (i) $\frac{54}{125}$ (ii) $\frac{27}{125}$

3. (a) 0.0515 (b) 0.6634

4. (a) $n = 8$ (b) 0.9747

Self-Exercise 5.7

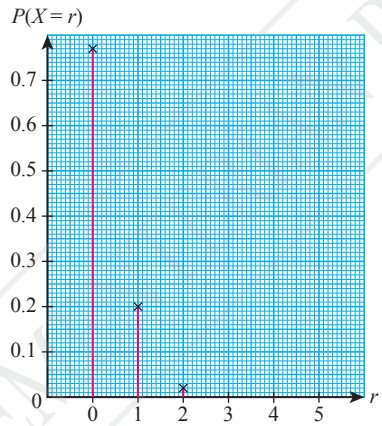
1. (a) 0.0951 (b) 0.6809

2. (a) 0.1379 (b) 28

3. (a) 0.9792 (b) 0.0565

4. (a)

$X = r$	$P(X = r)$
0	0.7738
1	0.2036
2	0.0214
3	0.0011
4	0.00003
5	3.1×10^{-7}



(b) (i) 0.0214 (ii) 0.0226

5. (a) $X = \{0, 1, 2, 3, 4, 5\}$ (b) $\frac{2}{9}$
(c) 83.33%

6. (a) 0.0141 (b) 0.5267

Self-Exercise 5.8

1. $n = 56, p = \frac{4}{5}$ **2.** 48, 5.367

3. 4 000, 800, $20\sqrt{2}$ **4.** 600, $4\sqrt{15}$

Self-Exercise 5.9

1. (a) $\frac{1}{2\,187}$ (b) 0.3073 (c) 0.5706

2. (a) 0.6, 60 (b) 0.2322

3. (a) 9 (c) 3.139×10^{-4}

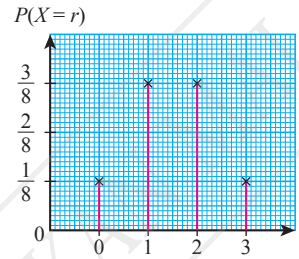
Formative Exercise 5.2

1.

$X = r$	$P(X = r)$
0	0.0625
1	0.2500
2	0.3750
3	0.2500
4	0.0625

2.

$X = r$	0	1	2	3
$P(X = r)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$



3. (a) 0.2725 (b) 2.423×10^{-4}

4. 5, 2.121

5. (a) $n = 25, p = \frac{1}{5}$ (b) 0.1358

6. (a) $\frac{2}{5}$, 4 (b) 0.2508

7. 10, 5

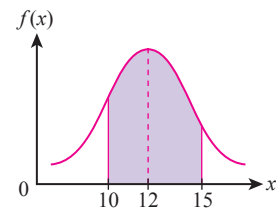
8. (a) $n = 4$ (b) 0.1808

9. (a) 12
(b) (i) 0.01 (ii) 1.359×10^{-3}

Self-Exercise 5.10

1. (a) 15 (b) $R: P(X < 12), Q: P(X > 18)$
(c) 0.2365, 0.5270

2. (a) 12
(b)



Self-Exercise 5.11

1. −0.75 **2.** 517.55

3. (a) −0.2 (b) 0.144 kg

4. 45, 10

Self-Exercise 5.12

1. $P\left(-\frac{14}{9} < Z < \frac{5}{9}\right)$

2. (a) 0.7046 (b) 0.8671 (c) 0.3359 (d) 0.4764

3. 0.0157, 0.8606, 0.5664, 0.2876, 0.2286, 0.3785, 0.821, −0.984, −0.107, 0.471, 0.729

4. (a) 0.274 (b) 0.116

5. 1.657 **6.** 1.333

7. 16.98 **8.** 52.73, 11.96

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Self-Exercise 5.13

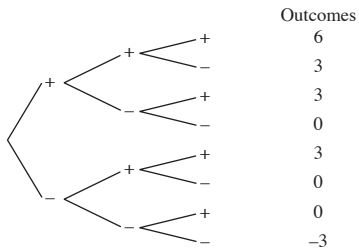
1. (a) 0.5 (b) 188.4
2. 24.34
3. (a) 0.6915 (b) 311
4. (a) 5 (b) 47
5. 52.07, 17.89
6. (a) 0.8383 (b) 100

Formative Exercise 5.3

1. –1.001
2. (a) 1.1 (b) 0.4649
3. 0.1244
4. (a) 0.4950 (b) 2.898 kg
5. (a) 16.48 (b) 1 008
6. (a) 74 (b) 63.06

Summative Exercise

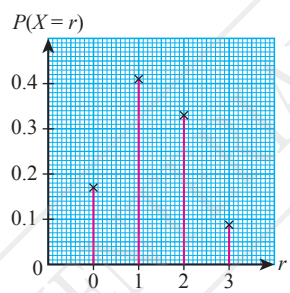
1. $X = \{2, 4, 6, 8, 10, 12\}$
2. (a) $\frac{1}{6}$ (b) $\frac{1}{2}$
3. (a)



(b) $X = \{-3, 0, 3, 6\}$

4. (b)

$X = r$	0	1	2	3
$P(X = r)$	0.1664	0.4084	0.3341	0.0911

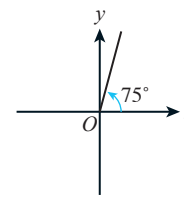


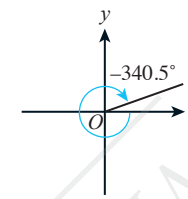
5. (a) 0.3110 (b) 0.0410 (c) 0.5443
6. (a) 0.1239 (b) 0.5941
7. (a) 0.1672 (b) 0.2318
8. 7, 2.366
9. (a) $\frac{3}{5}$ (b) $\frac{9}{25}$
10. (a) 0.5332 (b) 0.2315 (c) 0.5497 (d) 0.0995
(e) 44.5 (f) 59.42 (g) 57.37 (h) –39.61
11. (a) 15 (b) 112.47
12. (a) 352 (b) 77.34 kg
13. (a) 0.1266 (b) 498 (c) 179

CHAPTER 6 TRIGONOMETRIC FUNCTIONS

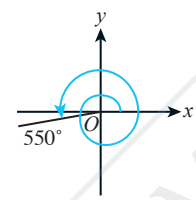
Self-Exercise 6.1

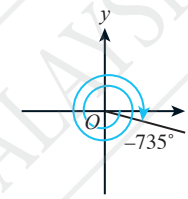
1. (a) 5.064 rad (b) –6.273 rad
(c) 10.82 rad (d) –13.79 rad
2. (a) 74.48° (b) 186.21°
(c) –486° (d) 585°
3. (a) Quadrant I (b) Quadrant I



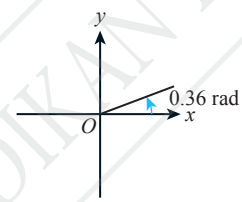


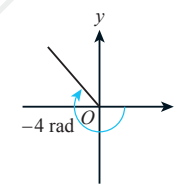
(c) Quadrant III (d) Quadrant IV



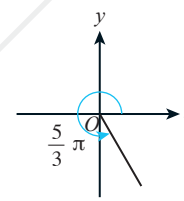


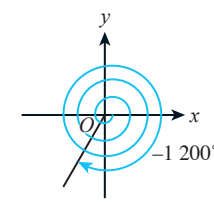
(e) Quadrant I (f) Quadrant II





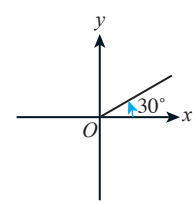
(g) Quadrant IV (h) Quadrant III

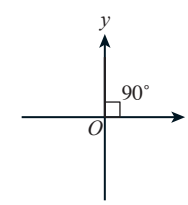


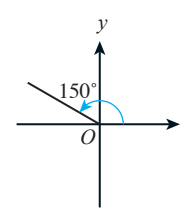


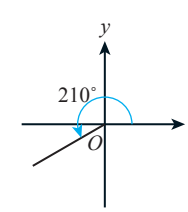
Formative Exercise 6.1

1. 0° = 0 rad, 30° = 0.5236 rad, 90° = 1.571 rad
150° = 2.618 rad, 210° = 3.665 rad,
270° = 4.712 rad, 330° = 5.760 rad,
360° = 6.283 rad

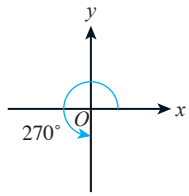


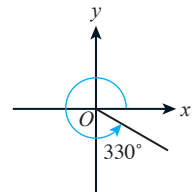






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Self-Exercise 6.2

1. (a) $\sqrt{\frac{23}{2}}$ (b) $\frac{2}{25}$ (c) $\frac{\sqrt{46}-2}{25}$

2. (a) $\frac{2}{\sqrt{13}}$ (b) $\frac{9}{13}$ (c) $\frac{3}{2}$

(d) $\frac{\sqrt{13}}{2}$ (e) $\frac{23}{3(6-\sqrt{13})}$

3. (a) 36° (b) 84° 42′ 46″ (c) $\frac{3}{10}\pi$

4. (a) 0.839 (b) 1.539 (c) 1.835

Self-Exercise 6.3

1. (a) −0.2549 (b) −3.7321 (c) 1.1511
(d) 1.3054

2. (a) $-\frac{1}{2}$ (b) $-\sqrt{3}$ (c) $\sqrt{3}$
(d) $-\frac{1}{2}$ (e) 1 (f) 2

3. (a) 25° (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{3}$ (d) 10°

4. (a) $-\frac{2}{\sqrt{3}}$ (b) $-\frac{2}{\sqrt{3}}$ (c) −1
(d) 0 (e) 6 (f) −1

Formative Exercise 6.2

1. (a) $\frac{1}{3t}$ (b) $\frac{\sqrt{1+9t^2}}{3t}$ (c) $\frac{\sqrt{1+9t^2}}{3t}$

2. (a) $\frac{1}{3}$ (b) 3 (c) $\frac{3}{\sqrt{10}}$

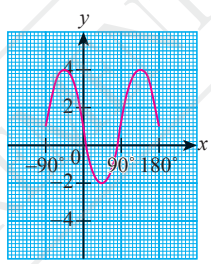
3. (a) $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ (b) $\frac{2}{\sqrt{3}}$
(c) $\frac{5}{2}$ (d) 6

4. (a) 0.6820 (b) 1.095 (c) 0.9656 (d) 3.732

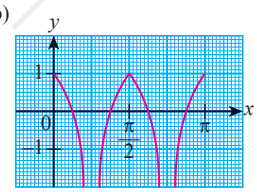
5. (a) $\frac{\sqrt{2}}{2}$ (b) $-\sqrt{2}$ (c) 1 (d) $-\sqrt{2}$

Self-Exercise 6.4

1. (a)



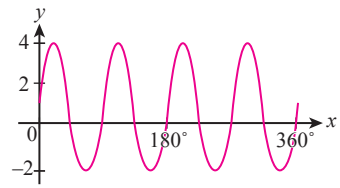
(b)



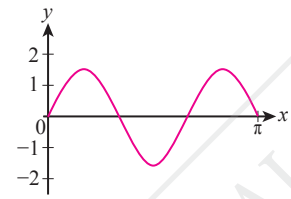
2. (a) $y = \tan x + 3$ (b) $y = 2\cos 3x - 1$

3. (a) $A = 3, B = 4, C = 1$

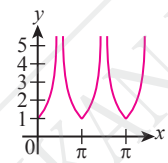
(b)



4. $y = \frac{3}{2}\sin 3x$: $\frac{3}{2}$, 3, 0

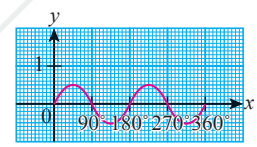


$y = |\tan 2x| + 1$: None, 4, 1

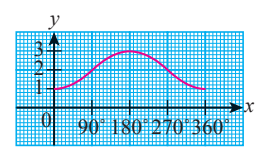


Self-Exercise 6.5

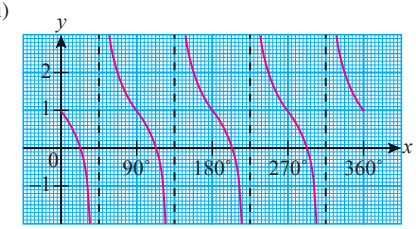
1. (a) (i)



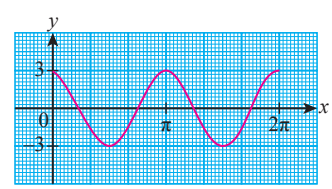
(ii)



(iii)

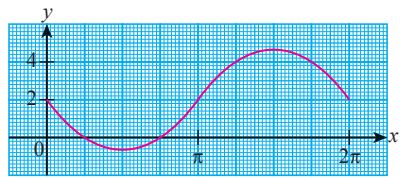


(b) (i)

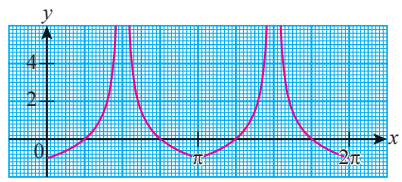


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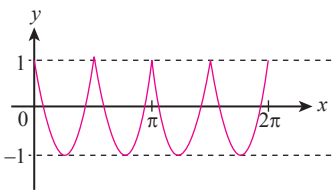
(ii)



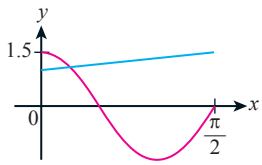
(iii)



2.

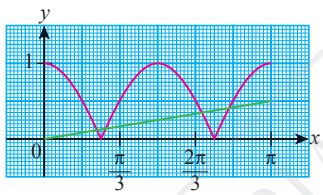


3.



Number of solutions = 1

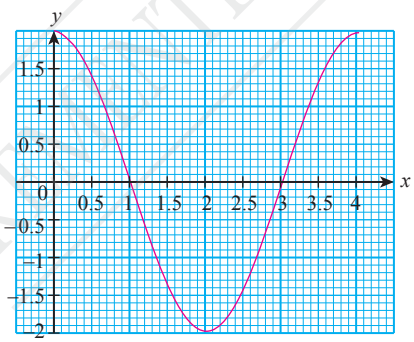
4.



Number of solutions = 4

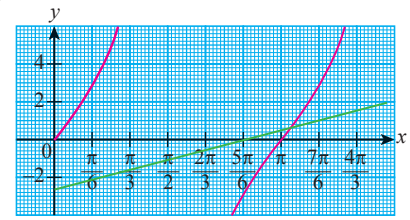
Formative Exercise 6.3

1.



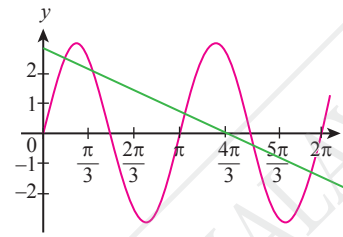
$x = 1.0, 3.0$

2.



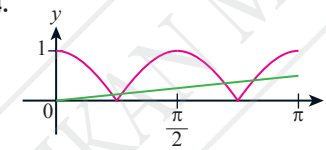
$x = 3.30$ radian

3.



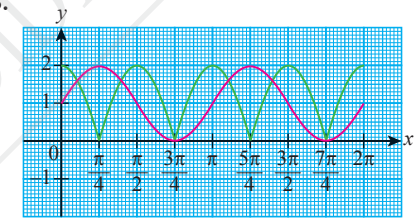
Number of solutions = 5

4.



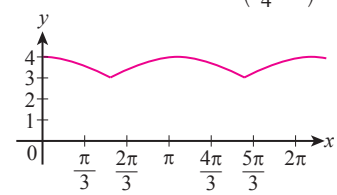
Number of solutions = 4

5.



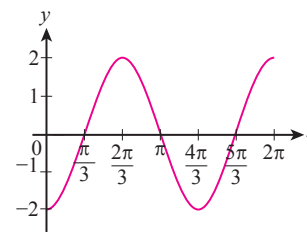
Intersection points: (0.322, 1.6), (1.249,1.6), $\left(\frac{3\pi}{4}, 0\right)$, (3.463,1.6), (4.391,1.6), $\left(\frac{7\pi}{4}, 0\right)$

6.



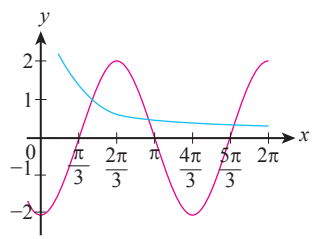
$k < 3, k > 4$

7. (a)



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(b)



Number of solutions = 3

Self-Exercise 6.6

1. (a) 1 (b) 1 (c) $\frac{1}{2}$ (d) 1

2. (a) $\frac{1}{m^2}$ (b) $1 - m^2$ (c) $\frac{m^2}{1 - m^2}$

3. $\sin\theta = \frac{3}{\sqrt{10}}$; $\cos\theta = \frac{1}{\sqrt{10}}$

4. (a) $\frac{p^2}{q^2}$ (b) $\frac{q^2 - p^2}{p^2}$ (c) $\frac{-p^2}{q^2 - p^2}$

Formative Exercise 6.4

1. (a) $p - 1$ (b) $\frac{1}{p}$ (c) $\frac{p - 1}{p}$

2. (a) 1 (b) –1 (c) 4 (d) 2

4. (b) 1.5626

Self-Exercise 6.8

2. (a) $\frac{\sqrt{6} - \sqrt{2}}{4}$ (b) $\frac{4}{\sqrt{6} + \sqrt{2}}$ (c) $\frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

3. (a) $-\frac{33}{65}$ (b) $-\frac{16}{65}$ (c) $-\frac{56}{33}$

Self-Exercise 6.9

1. (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $-\sqrt{3}$

4. (a) $\frac{25}{24}$ (b) $\frac{169}{119}$ (c) $\frac{1}{\sqrt{5}}$ (d) 5

Formative Exercise 6.5

1. $\frac{4}{3}$

3. (a) $\frac{416}{425}$ (b) $\frac{425}{297}$ (c) $-\frac{297}{304}$
(d) $-\frac{289}{161}$ (e) $-\frac{3}{\sqrt{34}}$

5. (a) $\frac{2t}{1 + t^2}$ (b) $\frac{1 - t^2}{1 + t^2}$ (c) $\frac{2t}{1 - t^2}$
(d) $\frac{\sqrt{1 + t^2} - 1}{2\sqrt{1 + t^2}}$ (e) $\frac{1 + \sqrt{1 + t^2}}{2\sqrt{1 + t^2}}$

Self-Exercise 6.10

1. (a) $x = 102.8°, 167.2°, 282.8°, 347.2°$
(b) $x = 10°, 130°, 190°, 310°$
(c) $x = 198°$
(d) $x = 0°, 44.42°, 180°, 315.58°, 360°$
(e) $x = 90°, 199.47°, 340.53°$
(f) $x = 150°, 330°$
(g) $x = 199.47°, 340.53°$
(h) $x = 0°, 80.41°, 180°, 279.59°, 360°$
(i) $x = 16.10°, 196.10°$

2. (a) $x = \frac{7}{12}\pi, \frac{3}{4}\pi, \frac{19}{12}\pi, \frac{7}{4}\pi$
(b) $y = 0$ rad , 0.2677π rad, π rad, 1.732π rad and 2π rad
(c) $z = \frac{1}{6}\pi$ rad, $\frac{5}{6}\pi$ rad (d) $A = \frac{1}{8}\pi, \frac{5}{8}\pi, \frac{9}{8}\pi, \frac{13}{8}\pi$
(e) $B = \frac{1}{12}\pi, \frac{5}{12}\pi, \frac{13}{12}\pi, \frac{17}{12}\pi$
(f) $x = \frac{13}{12}\pi, \frac{17}{12}\pi, \frac{25}{12}\pi, \frac{29}{12}\pi$

Self-Exercise 6.11

1. 550 kmh^{-1} **2.** 0.7071, –0.7071

3. (a) 1.5 (b) 0.8 (c) 0.3182
$\alpha = 38.66°$, $\beta = 17.65°$, $\angle BAC = 33.69°$, $\angle ADB = 128.66°$, $\angle BDC = 51.34°$, $BD = 12.81$ cm, $AB = 18.03$ cm

Formative Exercise 6.6

1. (a) $x = 130°, 250°$
(b) $64.27°, 140.13°, 219.87°, 295.73°$
(c) $126.87°, 306.87°$

2. (a) $A = 0, \frac{1}{6}\pi, \frac{1}{2}\pi, \frac{5}{6}\pi, \pi$
(b) $A = 0$ rad, 0.2852π rad, π rad

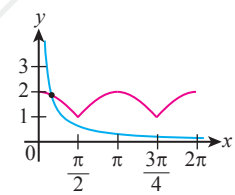
3. $\theta = 60°, 120°, 240°, 300°$

5. (a) $-\frac{8}{17}$ (b) $-\frac{240}{289}$ (c) $\frac{240}{161}$

6. (a) $\sin\angle CAD = \frac{24\sqrt{3} - 7}{50}$, $\cos\angle CAD = \frac{24 + 7\sqrt{3}}{50}$,
$\tan\angle CAD = \frac{24\sqrt{3} - 7}{24 + 7\sqrt{3}}$
(b) $AC = 25$ m, $AD = 48$ m

8. (a) $\frac{\sqrt{t^2 - 1}}{t}$ (b) $-\frac{\sqrt{t^2 - 1}}{t}$ (c) $-\sqrt{t^2 - 1}$

9. (a) $1 \leqslant f(x) \leqslant 2$
(b)



Number of solutions = 1

Summative Exercise

1. (a) $0 \leqslant x \leqslant 2\pi$ (b) $-\pi \leqslant x \leqslant \frac{\pi}{2}$ (c) $\frac{3}{2}\pi \leqslant x \leqslant 4\pi$

2. (a) $0 < x < \frac{\pi}{2}$ (b) $\frac{\pi}{2} < x < \pi$ (c) $\pi < x < 2\pi$

3. (a) $41.30°, 138.70°, 221.30°, 318.70°$
(b) $63.90°, 116.10°, 243.90°, 296.10°$
(c) $41.36°, 138.64°, 221.36°, 318.64°$

4. (a) $-\frac{\sqrt{3}}{2}$ (b) $-\sqrt{3}$ (c) $\frac{2}{\sqrt{3}}$
(d) ∞ (e) –1 (f) $-\frac{1}{2}$

5. (a) $\frac{56}{65}, \frac{63}{16}$ (b) $\frac{56}{65}, -\frac{63}{16}$ (c) $\frac{56}{65}, -\frac{63}{16}$

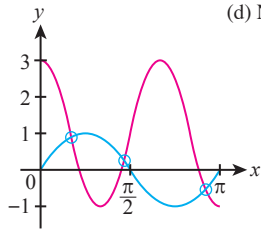
6.

Graph	Equation	Number of cycles	Period	Class interval
I	$y = \cos x$	1	2π	$\frac{\pi}{2}$
II	$y = \cos 2x$	2	π	$\frac{\pi}{4}$
III	$y = \cos \frac{1}{2}x$	$\frac{1}{2}$	4π	π

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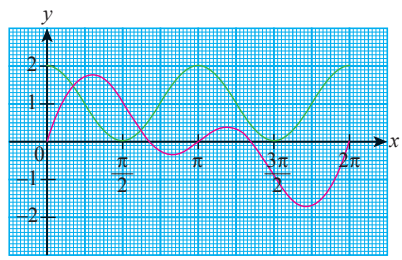
7. (a) π (b) 2, 3, –1

(c) (d) Number of solutions = 3



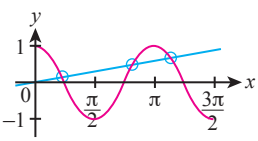
11. (a) $0, \frac{2}{3}\pi, \pi, \frac{4}{3}\pi, 2\pi$

(b) 2, π



(c) Number of solutions = 2

12. (b), (c)



Number of solutions = 3

13. (a) (i) $x = 60°, 240°$ (ii) $x = 7.063°, 187.063°$

(iii) $x = 48.43°, 228.43°$

(b) (i) $x = 0.3102$ rad, 3.452 rad

(ii) $x = 0.4637$ rad, 1.892 rad, 3.605 rad, 5.034 rad

(iii) $x = \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$

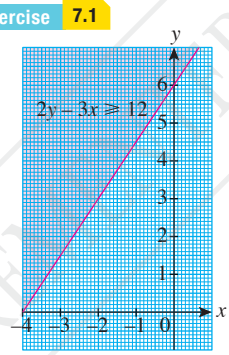
14. (a) 9.780 ms^{-2} (b) 9.8321 ms^{-2}

16. (a) $\cos x \sin x$ (b) $\sec x \operatorname{cosec} x$ (c) $\cos^2 x - \sin^2 x$

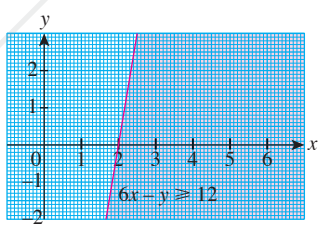
CHAPTER 7 LINEAR PROGRAMMING

Self-Exercise 7.1

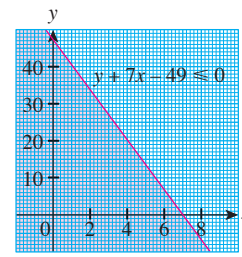
1. (a)



(b)



(c)



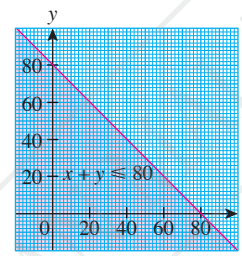
2. (a) $y \leqslant 3x$ (b) $x + y \leqslant 80$ (c) $y \geqslant 10$

3. (a) Area of the land is 80 hectares, 360 workers and the capital is RM24 000.

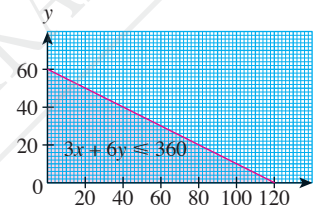
(b) (i) $x + y \leqslant 80$ (ii) $3x + 6y \leqslant 360$

(iii) $800x + 300y \geqslant 24\,000$

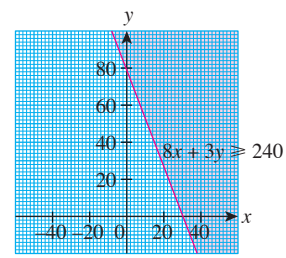
(c) (i)



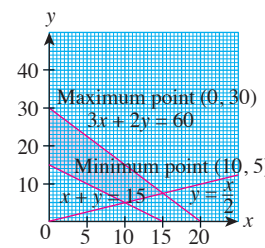
(ii)



(iii)



4. (a), (b)



(c) (i) 60 (ii) 20

Formative Exercise 7.1

1. (a) $y > x - 1$ (b) $y < 5x + 1$

2. I: $x + y \leqslant 100$, II: $y \leqslant 4x$, III: $y - x \geqslant 5$

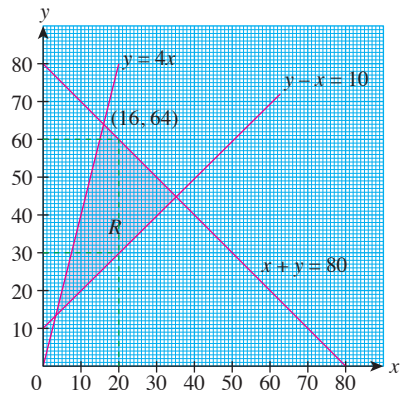
3. $y \leqslant 3x$, $y \leqslant x + 50$, $x + y \leqslant 1\,000$

Self-Exercise 7.2

1. (a) I: $x + y \leqslant 80$, II: $y \leqslant 4x$, III: $y - x \geqslant 10$

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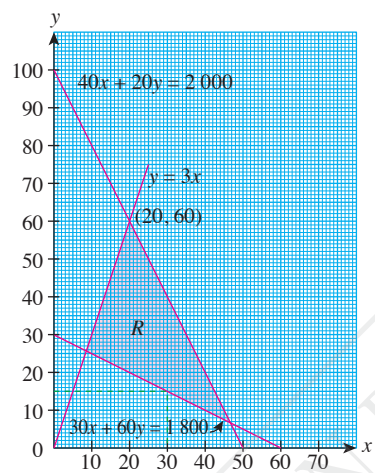
(b)



(c) (i) $30 \leqslant y \leqslant 60$ (ii) RM5 440

2. (a) $40x + 20y \leqslant 2\,000$ or its equivalent,
$30x + 60y \geqslant 1\,800$ or its equivalent,
$y \leqslant 3x$ or its equivalent

(b)

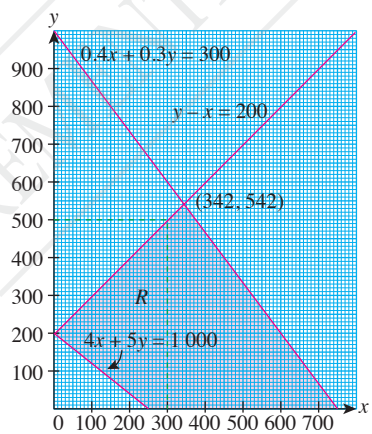


(c) (i) 15 (ii) RM21 000

Formative Exercise 7.2

1. (a) $4x + 5y \geqslant 1\,000$ or its equivalent,
$0.4x + 0.3y \leqslant 300$ or its equivalent,
$y - x \leqslant 200$ or its equivalent

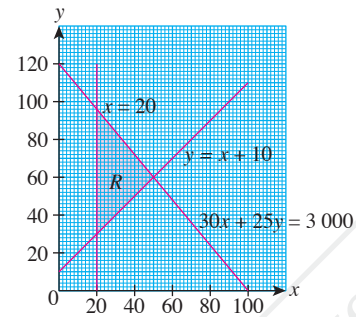
(b)



(c) (i) 500 (ii) RM2 497.80

2. (a) $30x + 25y \leqslant 3\,000$, $x \geqslant 20$, $y \geqslant x + 10$

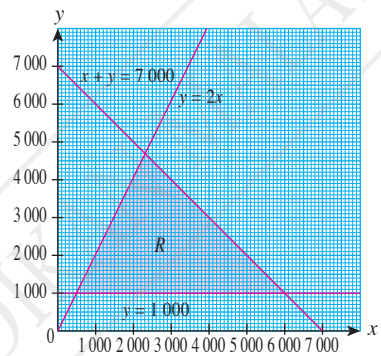
(b)



(c) RM1 350

3. (a) $x + y \leqslant 7\,000$, $y \leqslant 2x$, $y \geqslant 1\,000$

(b)

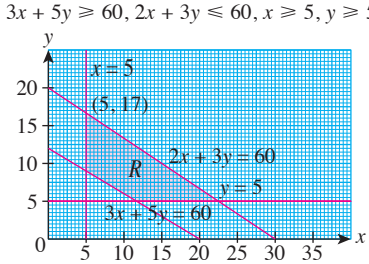


(c) (i) 5 000 litres (ii) RM330 000

Summative Exercise

1. (a) $3x + 5y \geqslant 60$, $2x + 3y \leqslant 60$, $x \geqslant 5$, $y \geqslant 5$

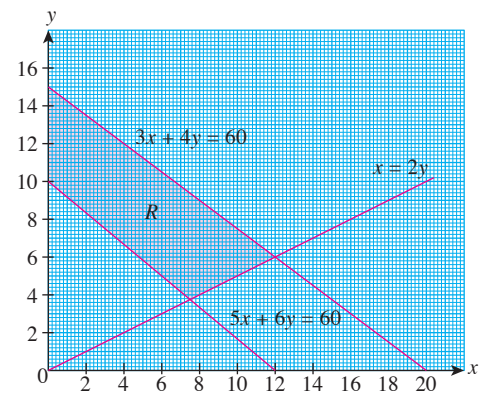
(b)



(c) (i) $x = 5$, $y = 17$ (ii) RM1 560

2. (a) $5x + 6y \geqslant 60$, $3x + 4y \leqslant 60$, $x \leqslant 2y$

(b)

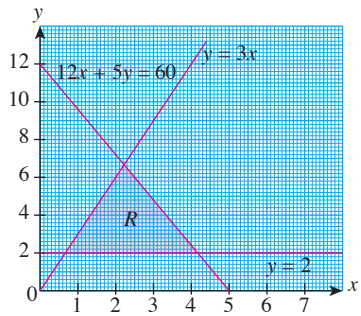


(c) RM324

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3. (a) $12x + 5y \leqslant 60$, $y \leqslant 3x$, $y \geqslant 2$

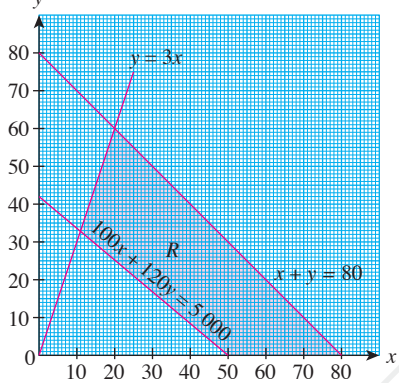
(b)



(c) (i) $1 \leqslant y \leqslant 4$ (ii) RM800

4. (a) $x + y \leqslant 80$, $\frac{x}{y} \geqslant \frac{1}{3}$ or $y \leqslant 3x$, $100x + 120y \geqslant 5\,000$

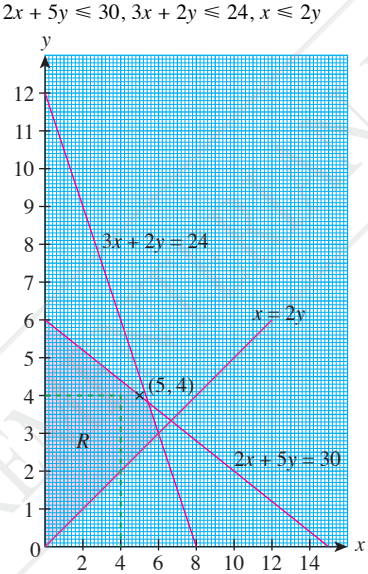
(b)



(c) (i) 33 (ii) RM2 300

5. (a) $2x + 5y \leqslant 30$, $3x + 2y \leqslant 24$, $x \leqslant 2y$

(b)



(c) (i) 4 (ii) RM2 000

CHAPTER 8 KINEMATICS OF LINEAR MOTION

Self-Exercise 8.1

1. (a) (i) –3 m (ii) –5 m

(b) (i) 3 s (ii) 4 s (c) $t > 3$

2. (a) –8 ms^{-1} (b) 1 s and 7 s (c) $1 < t < 7$

3. (a) –8 ms^{-2} (b) 2 s (c) $t < 2$

Self-Exercise 8.2

1. (a) 68 m (b) 111 m

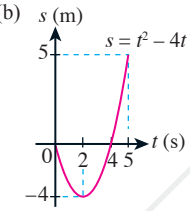
2. (a) (i) 8 m (ii) 45 m (b) 7 m

Formative Exercise 8.1

1. (a)

t (s)	1	2	3	4	5
s (m)	–3	–4	–3	0	5

(b) s (m)



(c) 4 s

2. (a) 2 (b) (i) 14 m (ii) 35 m

3. 0, 320 m **4.** 9 ms^{-1}, 23 ms^{-1}

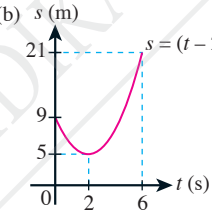
5. (a) 4 ms^{-2} (b) $t > 2$

6. (a) 21 m (b) 55 m

7. (a)

Time, t (s)	0	1	2	3	4	5	6
Displacement, s (m)	9	6	5	6	9	14	21

(b) s (m)



(c) 20 m

Self-Exercise 8.3

1. (a) $v = 4 - 8t + 3t^2$ (b) $v = 16 - 2t$

(c) $v = 6t^2 - 8t + 2$ (d) $v = 27t^2 + 24t^3 + 5t^4$

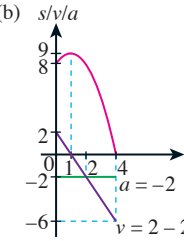
(e) $v = 6t^2 - 18t - 5$ (f) $v = t^2 - 6t + 5$

2. (a) $a = 2t - 1$ (b) $a = 6t - 10$

(c) $a = -12t$ (d) $a = 18t - 30$

(e) $a = 6t + \frac{1}{t^2}$ (f) $a = 18t^2 + \frac{8}{t^3}$

3. (a) $v = 2 - 2t$, $a = -2$ (b) $s/v/a$



Self-Exercise 8.4

1. (a) (i) –2 ms^{-1} (ii) 5 ms^{-1} (iii) 21 ms^{-1}

(b) (i) $\frac{1}{2}$ second (ii) 2 seconds (iii) 3 seconds

2. (a) 8 ms^{-1} (b) $\frac{2}{3}$ second, 1 second

(c) $0 \leqslant t < \frac{2}{3}$ or $t > 1$

Self-Exercise 8.5

1. (a) 8 ms^{-2} (b) –8 ms^{-2} (c) 4 seconds

2. (a) 1 second (b) $t < 1$

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Formative Exercise 8.2

1. (a) $-2\ \text{ms}^{-1}$ (b) 3 seconds (c) 4 m
(d) 10 m (e) $t > 1$

2. (a) $h = \frac{1}{2}$, $k = 1$
(b) (i) $1\ \text{ms}^{-1}$ (ii) $0\ \text{ms}^{-1}$ (iii) $-1.5\ \text{ms}^{-1}$

3. (a) $v = 3t^2 - 10t - 8$, $a = 6t - 10$
(b) $-11\ \text{ms}^{-1}$, $8\ \text{ms}^{-2}$ (c) 4 s
(d) 1 s, 6 s (e) 84 m

Self-Exercise 8.6

1. (a) $10\ \text{ms}^{-1}$ (b) $2\ \text{ms}^{-1}$
2. (a) $-13\ \text{ms}^{-1}$ (b) $-2\ \text{ms}^{-1}$
3. (a) $2 < t < 6$ (b) $-12\ \text{ms}^{-1}$
4. (a) $32\ \text{cms}^{-1}$ (b) $36\ \text{cms}^{-1}$ (c) 1.5 s (d) 9 s

Self-Exercise 8.7

1. (a) $27\frac{1}{2}$ m (b) 14 m
2. (a) -10 m (b) -16 m
3. (a) 48 m (b) 8 m
4. (a) $s = \frac{4}{3}t^3 + 3t^2 - 18t$, $v = 4t^2 - 6t - 18$
(c) 9 km

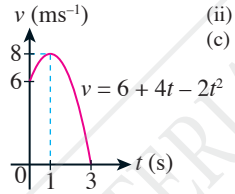
Formative Exercise 8.3

1. (a) $42\ \text{ms}^{-1}$ (b) 35 m on the right O
2. (a) $24\ \text{ms}^{-2}$ (b) 2 s (c) 6 s
3. (a) $m = -10$, $n = 4$ (b) $-24.5\ \text{ms}^{-1}$ (c) 189 m
4. (a) $-\frac{27}{2}$ m (b) $t < \frac{5}{4}$ (c) 63 m
5. (a) $18\ \text{ms}^{-1}$ (b) $t = 0$ s, 6 s (c) $\frac{40}{3}$ m
6. (a) $\frac{25}{2}$ s (b) $\frac{75}{16}\ \text{ms}^{-1}$ (c) $\frac{625}{8}$ m

Self-Exercise 8.8

1. (a) 2 seconds (b) $s = 20t - 5t^2$
(c) (i) 20 m (ii) 4 seconds

2. (a) $8\ \text{ms}^{-1}$
(b) (i) $v\ (\text{ms}^{-1})$ (ii) 18 m
(c) 17 m

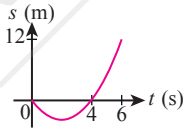


3. (a) $m = 12.5$, $n = -12.5$ (b) $-3.125\ \text{kmh}^{-1}$
(c) $\frac{125}{12}$ km

4. (a) $20\ \text{ms}^{-2}$ (b) 9 m

Formative Exercise 8.4

1. (a) $56\ \text{ms}^{-1}$ (b) 104 m
2. (a) $8\ \text{ms}^{-1}$ (b) $40\ \text{ms}^{-1}$
3. (a) $t < 2$ (b) No (c) 20 m
(d) s (m)



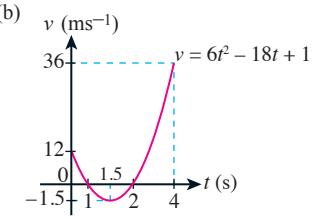
4. (a) 47 m (b) $-\frac{4}{3}\ \text{ms}^{-1}$
(c) $\frac{2}{3} < t < 2$ (d) $\frac{86}{27}$ m

Summative Exercise

1. (a) 208 m (b) $48\ \text{ms}^{-1}$ (c) $-12\ \text{ms}^{-2}$ (d) $t = 3$ s, 5 s

2. (a) -7 m (b) $-12\ \text{ms}^{-1}$ (c) $6\ \text{ms}^{-2}$

3. (a) (i) $12\ \text{mmin}^{-1}$ (ii) $12\ \text{mmin}^{-1}$
(iii) $6\ \text{mmin}^{-2}$ (iv) 149 m
(b)



4. (a) $v = 10t$, $s = 5t^2$
(b) The particle is at point X after 0.5 second with a velocity of $5\ \text{ms}^{-1}$.

5. (a) $8\ \text{ms}^{-1}$ (b) -10 m

6. (a) $t = 1$ s, 3 s (b) $69\frac{1}{3}$ m

7. (a) $m = 5$, $n = -20$ (b) $s = \frac{5}{6}t^3 - 10t^2 + 30t$
(c) 6 seconds (d) $\frac{35}{6}$ m

8. (a) $-10\ \text{ms}^{-1}$ (b) $14\ \text{ms}^{-2}$

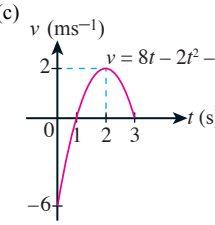
9. (a) $t = 4$ (b) $\frac{113}{6}$ m
(c) The car reverses for 4 seconds and then moves forward.

10. (b) -43 m

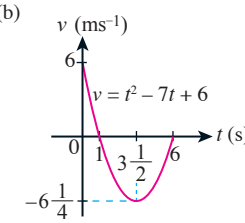
11. (a) $v = (3t^2 - 3)\ \text{ms}^{-1}$, $a = 6t\ \text{ms}^{-2}$
(b) The particle moves to the left with initial velocity of $-3\ \text{ms}^{-1}$ and zero acceleration. For $t = 2$, the particle moves to the right with velocity of $9\ \text{ms}^{-1}$ and experiences acceleration of $12\ \text{ms}^{-2}$.
(c) $t > 1$

12. (a) $h = 3$, $k = -9$ (b) 4.5 s
(c) $18\ \text{ms}^{-2}$ (d) 14.5 m

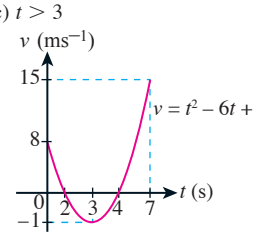
13. (a) $2\ \text{ms}^{-1}$ (b) 3 seconds
(c) (d) $\frac{16}{3}$ m



14. (a) (i) $6\ \text{cms}^{-1}$ (ii) $1 < t < 6$ (iii) $t > \frac{7}{2}$
(b)



15. (a) $-1\ \text{ms}^{-1}$ (b) $\frac{4}{3}$ m
(c) $t > 3$



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Glossary

Acceleration (*Pecutan*) Rate of change of velocity.

Arc of a circle (*Lengkok bulatan*) Arc is part of the circumference of the circle.

Binomial distribution (*Taburan binomial*) The probability distribution involving n Bernoulli trials which are similar or identical where the possibility of 'success' is constant in every trial and every trial is independent of each other.

Binomial experiment (*Eksperimen binomial*) Composed of n Bernoulli trials which are independent but similar. Each trial has only two outcomes, which are 'success' and 'failure'.

Chord (*Perentas*) A straight line connecting any two points on the circumference of the circle.

Circumference of a circle (*Lilitan bulatan*) Perimeter for a circle.

Combination (*Gabungan*) A selection of all or part of a set of objects, regardless of their orders of the selected objects.

Complementary angle (*Sudut pelengkap*) Angle A is the complementary angle of angle B if $A + B = 90°$.

Composite function (*Fungsi gubahan*) A function that combines two or more functions.

Constant velocity (*Halaju malar*) The velocity of linear motion of an object that is not changing.

Constraint (*Kekangan*) Limitations within a situation like a lack of raw materials, capital, operating time and so on.

Definite integral (*Kamiran tentu*) An integration whose value is fixed by a certain range of values.

Event (*Peristiwa*) Set of possible outcomes for an experiment. An event is a subset of sample space.

Factorial (*Faktorial*) n objects can be arranged in $n(n-1)(n-2)\ldots(3)(2)(1)$ ways. This product can be represented by the symbol $n!$ which is called a n factorial.

Feasible region (*Rantau tersaur*) A region that satisfies all the mathematical model requirements for a situation.

Generated volume (*Isi padu janaan*) The volume of an object formed when a shaded region rotates on an axis, which can be the x-axis or the y-axis.

Gradient of tangent (*Kecerunan tangen*) Gradient of a straight line that touches a curve at only one point.

Indefinite integral (*Kamiran tak tentu*) Integration without limits.

Instantaneous acceleration (*Pecutan seketika*) Rate of change in velocity at a particular time.

Integration (*Kamiran*) A concept in calculus which is the inverse of differentiation.

Kinematic of linear motion (*Kinematik gerakan linear*) Kinematic means the movement of an object represented by a straight line in words, diagrams, numbers, graphs and equations.

Limit (*Had*) The value of a function when a variable approaches a certain value.

Maximum displacement (*Sesaran maksimum*) Distance between the end point and the starting point in a straight line when the velocity is zero.

Negative angle (*Sudut negatif*) The angle formed by rotating a straight line at an origin clockwise from the positive x-axis.

Normal (*Normal*) A perpendicular straight line to its tangent line.

Normal distribution (*Taburan normal*) A continuous random variable and is one of the most important distributions in statistics because it represents many natural phenomena. The distribution graph is bell-shaped.

Objective function (*Fungsi objektif*) A function used to determine the optimal value.

Positive angle (*Sudut positif*) An angle formed by rotating a straight line at an origin anticlockwise from the positive x-axis.

Radian (*Radian*) The unit used to measure the size of an angle in circular measure.

Radius (*Jejari*) A straight line from the centre of the circle to any point on the circumference of the circle.

Random variables (*Pemboleh ubah rawak*) A random variable is a variable whose value is numeric and from a random phenomenon.

Segment (*Tembereng*) Region that is bounded by a curve and a chord.

Standard normal distribution (*Taburan normal piawai*) A normal distribution with a mean of 0 and a standard deviation of 1.

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